STABILITY OF THE POSITIVE MASS THEOREM FOR ROTATIONALLY SYMMETRIC RIEMANNIAN MANIFOLDS

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Abstract. We study the stability of the Positive Mass Theorem using the Intrinsic Flat Distance. In particular we consider the class of complete asymptotically flat rotationally symmetric Riemannian manifolds with nonnegative scalar curvature and no interior closed minimal surfaces whose boundaries are either outermost minimal hypersurfaces or are empty. We prove that a sequence of these manifolds whose ADM masses converge to zero must converge to Euclidean space in the pointed Intrinsic Flat sense. In fact we provide explicit bounds on the Intrinsic Flat Distance between annular regions in the manifold and annular regions in Euclidean space by constructing an explicit filling manifold and estimating its volume. In addition, we include a variety of propositions that can be used to estimate the Intrinsic Flat distance between Riemannian manifolds without rotationally symmetry. Conjectures regarding the Intrinsic Flat stability of the Positive Mass Theorem in the general case are proposed in the final section.

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1. Introduction

The Positive Mass Theorem states that any complete asymptotically flat manifold of nonnegative scalar curvature has nonnegative ADM mass. Furthermore, if the ADM mass is zero, then the manifold must be Euclidean space. The second statement may be thought of as a rigidity theorem, and it is natural to consider the *stability* of this rigidity statement. That is, if the ADM mass is small, in what sense can we say that the manifold is "close" to Euclidean space? This is known to be a subtle question for many reasons.

The ADM mass was defined by Arnowitt-Deser-Misner in [2] and the Positive Mass Theorem was first proven in the rotationally symmetric case by physicists Jang, Leibovitz and Misner in [17] [21] [23]. The general three dimensional case was proven by Schoen-Yau, and later by Witten using spinors [26] [32]. Schoen and Yau's proof generalizes to dimensions < 8 (see [27]) using Bartnik's higher dimensional ADM mass [3], while Witten's proof holds on all spin manifolds (c.f. [27]).

The problem of stability for the Positive Mass Theorem has been studied by the first author in [19], by Finster with Bray and Kath in [5], [9] [8] and by Corvino in [6]. The work of Finster and his collaborators mainly focuses on using the ADM mass to obtain L^2 bounds on curvature. Corvino proves that with uniform bounds on sectional curvature, a manifold with small enough ADM mass is diffeomorphic to Euclidean space. The present work complements the results of [19]. That article dealt with convergence to Euclidean space outside some compact set. In this paper we tackle the much harder problem of trying to understand what happens inside the compact set. We place no assumptions on sectional curvature, so it is possible for the manifolds to have boundary inside the compact region. Because we expect the general problem to be difficult, we focus on the simple case of rotationally symmetric manifolds and state a more general conjecture at the end of the paper.

One serious concern is that even if the ADM mass is small, there can be arbitrarily deep "gravity wells." See Figure 1 and Example 2.9. As the ADM mass approaches zero, these deep gravity wells do not converge to Euclidean space in any conventional sense, including the pointed Gromov-Hausdorff sense. For this reason, we turn to the Intrinsic Flat Distance between Riemannian manifolds, a notion developed by the second author and S. Wenger which can be controlled using volumes and filling volumes [30].

The Intrinsic Flat Distance is defined and studied in [30] by applying sophisticated ideas of Ambrosio-Kirchheim [1] extending earlier work of Federer-Fleming [7] and Whiney[31]. While the definition of the Intrinsic Flat Distance involves abstract metric spaces and geometric measure theory, for the present work we can restrict our attention to Riemannian manifolds. Given two compact orientable Riemannian manifolds M_1^m and M_2^m with boundary, and metric isometric embeddings $\psi_i : M_i \to Z$ into some Riemannian manifold (possibly piecewise smooth with corners), Z, an upper bound for the Intrinsic Flat Distance is attained as follows:

$$d_{\mathcal{F}}(M_1^m, M_2^m) \le \mathrm{Vol}_{m+1}(B^{m+1}) + \mathrm{Vol}_m(A^m) \tag{1}$$

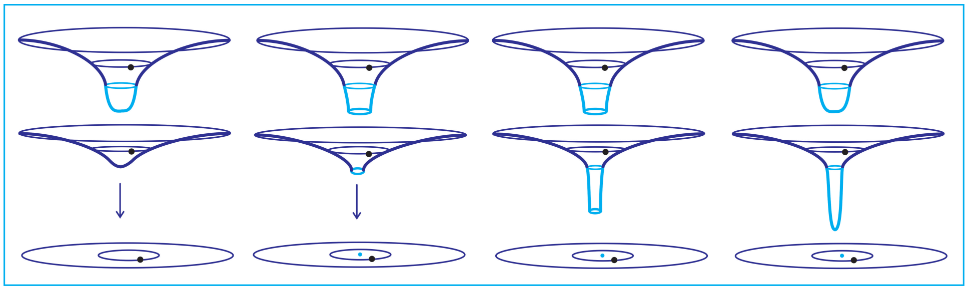

FIGURE 1. Four sequences of asymptotically flat manifolds of nonnegative scalar curvature whose ADM masses converge to zero.

where B^{m+1} is an oriented region in Z and A^m is defined so that

$$\int_{\psi_1(M_1)} \omega - \int_{\psi_2(M_2)} \omega = \int_{\partial B} \omega + \int_A \omega \tag{2}$$

for any m differential form ω on Z. We call B^{m+1} a *filling manifold* between M_1 and M_2 and A^m the *excess boundary*. A metric isometric embedding, $\psi : M \to Z$ is a map such that

$$d_Z(\psi(x), \psi(y)) = d_M(x, y) \qquad \forall x, y \in M. \tag{3}$$

This is significantly stronger than a Riemannian isometric embedding which preserves only the Riemannian structure and thus lengths of curves but not distances between points as in (3).

Our main results concern rotationally symmetric Riemannian manifolds of dimension 3 and up:

Definition 1.1. *Given $m \geq 3$, let* RotSym_m *be the class of complete m-dimensional rotationally symmetric Riemannian manifolds of nonnegative scalar curvature with no closed interior minimal hypersurfaces which either have no boundary or have a boundary which is a stable minimal hypersurface.*

This class of spaces includes the classical rotationally symmetric gravity wells and Schwarzschild spaces. The boundary, when it exists, is called the "apparent horizon" of a black hole. Nonnegative scalar curvature may be viewed as a physical notion of nonnegative mass density. The ADM mass of such a manifold exists when it is asymptotically flat and intuitively records the total mass of the space as a physical system. We review the scalar curvature and ADM mass of manifolds in this class within the paper.

The condition regarding minimal hypersurfaces is included here (just as it is in the Penrose Inequality) because complicated geometry can "hide" behind a minimal hypersurface without affecting the ADM mass (c.f. [11] [15]). Note that we need not explicitly assume asymptotic flatness here because finite ADM mass in RotSym_m implies asymptotic flatness.

Theorem 1.2. *Given any $\epsilon > 0$, $D > 0$, $A_0 > 0$, $m \in \mathbb{N}$ there exists a $\delta = \delta(\epsilon, D, A_0, m) > 0$ such that if $M^m \in \mathrm{RotSym}_m$ has ADM mass $\mathrm{m}_{\mathrm{ADM}}(M) < \delta$ and*

$\mathbb{E}^m$ is Euclidean space of the same dimension, then

$$d_{\mathcal{F}}(\,T_D(\Sigma_0) \subset M^m\,,\,T_D(\Sigma_0) \subset \mathbb{E}^m\,) < \epsilon. \tag{4}$$

where Σ_0 is the symmetric sphere of area $\mathrm{Vol}_{m-1}(\Sigma_0) = A_0$, and $T_D(\Sigma)$ is the tubular neighborhood of radius D around Σ_0.

See Remark 4.6 concerning the fact that the flat distance does not scale with the metric on the manifolds. In the proof precise estimates on $\delta(\epsilon, A_0, D, m)$ are provided.

Applying Ambrosio-Kirchheim's Slicing Theorem as in [29], we then have the following immediate corollary:

Corollary 1.3. *Let M_j^m be a sequence in RotSym_m. Fix an area A_0, and choose $p_j \in \Sigma_j$ to lie on the symmetric sphere $\Sigma_j \subset M_j^m$ of area $\mathrm{Vol}_{m-1}(\Sigma_j) = A_0$. If $\mathrm{m}_{\mathrm{ADM}}(M_j)$ converges to 0 then (M_j^m, p_j) converges to Euclidean space $(\mathbb{E}^m, 0)$ in the pointed intrinsic flat sense. That is, for almost any $D > 0$ there exists $D_j \to D$ such that $B_{p_j}(D_j) \subset M_j^m$ converges in the intrinsic flat sense to $B_0(D) \subset \mathbb{E}^m$.*

Throughout the paper, we provide techniques which can be used in a more general setting to bound the Intrinsic Flat Distance using Riemannian methods rather than Geometric Measure Theory. These may be applied to solve some of the open problems in our final section or even problems which do not involve scalar curvature.

In Section 2 we review rotationally symmetric manifolds with nonegative scalar curvature, the monotonicity of Hawking mass and the definition of ADM mass. There we describe a well known Riemannian isometric embedding of these manifolds as graphs in Euclidean space [Lemma 2.1] and review the monotonicity of Hawking mass [Theorem 2.2]. We control the diameter of the boundary in terms of the ADM mass [Lemma 2.4] and the slope of the graph [Lemma 2.5] in terms of the ADM mass. We conclude the section with classical rotationally symmetric examples including those depicted in Figure 1 [Examples 2.7 and 2.9].

In Section 3 we prove a variety of propositions about metric isometric embeddings and estimates on the intrinsic flat distance. This includes Theorem 3.1 using warped products to construct metric isometric embeddings and Theorem 3.2 regarding the construction of metric isometric embeddings from convex embeddings. The later theorem may be useful to those studying quasilocal mass. Theorem 3.3 provides a general method for constructing a metric isometric embedding from a Riemannian isometric embedding using an "embedding constant". This theorem is applied to bound the Intrinsic Flat Distance as a function of the embedding constant in Propositions 3.4 and 3.5. Theorem 3.6 provides a bound on the embedding constant when the Riemannian isometric embedding is a graph over a manifold with boundary. See also Remark 3.8.

In Section 4 we prove Theorem 1.2. See Figure 3 for a depiction of the explicit filling manifold and excess boundary used in the proof. Lemma 4.1 determines where to cut off a possibly deep well in the estimate. Then the earlier theorems and lemmas are applied to prove we have metric isometric embeddings and to estimate the volumes.

In Section 5 we review Gromov-Hausdorff convergence. We provide a new method for estimating the Gromov-Hausdorff distance using embedding constants, [Propositions 5.1 and 5.2] and apply them to construct explicit examples demonstrating that even with an assumption on rotational symmetry, the Positive Mass Theorem is not stable with respect to the Gromov-Hausdorff distance [Example 5.3] due to the existence of thin deep wells. This section closes with an example of a sequence of 3 dimensional manifolds with positive scalar curvature with no rotational symmetry whose ADM mass converges to 0 but has no subsequence converging in the Gromov-Hausdorff sense to any space due to the existence of an increasingly dense collection of wells [Example 5.6]. Nevertheless this sequence converges in the Intrinsic Flat sense to Euclidean space.

In Section 6 we discuss the general question of asymptotically flat Riemannian manifolds, M^m, of positive scalar curvature with no interior minimal surfaces that either have an outermost minimizing boundary or no boundary. We close the paper with conjectures and open problems concerning various subclasses of such manifolds and the stability of the Positive Mass Theorem for those subclasses. We hope that some of our more general theorems regarding the Intrinsic Flat Distance will prove useful to those attempting these problems.

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2. Positive Scalar Curvature, ADM Mass and Asymptotic Flatness

In the first subsection we briefly review the properties of the manifolds in RotSym_m and the key formulas defining their ADM mass. In the next subsection we embed the manifold into Euclidean space as a graph and review the Positive Mass Theorem and the monotonicity of the Hawking mass. In the third subsection we explore geometric implications of having a small ADM mass proving key lemmas which will be applied later to prove the stability of the positive mass theorem. We close with a subsection providing key rotationally symmetric examples.

2.1. **Setting.** In this paper we consider manifolds $(M^m, g) \in \mathrm{RotSym}_m$ defined in Definition 1.1. Since such a manifold is rotationally symmetric we can write its metric in geodesic coordinates, as $g = ds^2 + f(s)^2 g_0$ for some function $f : [0,\infty) \to [0,\infty)$ where g_0 is the standard metric on the $(m-1)$-sphere and s is either the distance from the pole, p_0, or from the boundary, ∂M.

Let Σ'_s be a level set of this distance function at a distance s from the pole or boundary. We then have the following formulae for the "area" and mean curvature

of Σ'_s:

$$\mathrm{A}(s) = \mathrm{Vol}_{m-1}(\Sigma_s) = \omega_{m-1} f^{m-1}(s) \tag{5}$$

$$\mathrm{H}(s) = \frac{(m-1)f'(s)}{f(s)} \tag{6}$$

Thus Σ'_s provide a CMC foliation of the manifold.

Let $r_{min} = f(0)$. When $\partial M = \emptyset$ then $f(0) = r_{min} = 0$ and $f(s) \geq f(0)$ by smoothness at the pole. When $\partial M \neq \emptyset$ the definition of RotSym states that ∂M is a stable minimal surface so $f(0) = r_{min} > 0$ and $f'(0) = 0$ and $f(s) \geq f(0)$ in that case as well.

The definition of RotSym also requires that M^m has no interior minimal surfaces, so by (6), we have

$$f'(s) \neq 0 \qquad \forall s \in (0, \infty]. \tag{7}$$

By the Mean Value Theorem, we see that

$$f'(s) > 0 \qquad \forall s \in (0, \infty]. \tag{8}$$

Thus $A(s)$ is increasing and we can uniquely define our rotationally symmetric constant mean curvature spheres

$$\Sigma_{\alpha_0} = \Sigma'_{s_0} \text{ such that } \mathrm{Vol}_{m-1}(\Sigma'_{s_0}) = \alpha_0. \tag{9}$$

Observe that intrinsically these are round spheres of diameter:

$$\mathrm{diam}(\Sigma_{\alpha_0}) = \pi f(s_0). \tag{10}$$

At a point $p \in \partial B_{p_0}(s)$ the scalar curvature is

$$\mathrm{R} = \frac{m-1}{f^2(s)} \Big((m-2)[1 - (f'(s))^2] - 2f(s)f''(s) \Big) > 0. \tag{11}$$

Recall the definition of the Hawking mass of a surface, Σ in three dimensional manifold:

$$\mathrm{m_H}(\Sigma) = \frac{1}{2}\left(\frac{A}{\omega_2}\right)\left(1 - \frac{1}{4\pi}\int_\Sigma \left(\frac{H}{2}\right)^2\right). \tag{12}$$

We define a natural Hawking mass function, $\mathrm{m_H}(s)$, for $M^m \in$ RotSym such that in dimension three $\mathrm{m_H}(s) = \mathrm{m_H}(\Sigma_s)$:

$$\mathrm{m_H}(s) = \frac{f^{m-2}(s)}{2}(1 - (f'(s))^2). \tag{13}$$

Applying (11), we see that

$$\mathrm{m'_H}(s) = \frac{f^{m-1}(s) f'(s)}{2(m-1)} \mathrm{R} \tag{14}$$

Since we are studying manifolds with $f'(s) > 0$ for $s \in (0, \infty)$ and $R \geq 0$, we have the monotonicity of the Hawking mass:

$$\mathrm{m'_H}(s) \geq 0. \tag{15}$$

Observe that when $\partial M \neq \emptyset$,

$$\mathrm{m_H}(0) = r_{min}^{m-2}/2. \tag{16}$$

This also holds true when $\partial M = \emptyset$, since $\mathrm{m_H}(0) = 0$.

We define the ADM mass of M^m is defined as the limit of the Hawking masses:

$$\mathrm{m_{ADM}}(M^m) = \lim_{s\to\infty} \mathrm{m_H}(s) \in [0, \infty]. \tag{17}$$

For rotationally symmetric manifolds, this agrees with the definition of the ADM mass in arbitrary dimensions.

Theorem 1.2 concerns manifolds whose ADM mass is finite and close to 0 which leads to almost equality in the following well known inequality:

$$0 \leq \mathrm{m_H}(0) \leq \mathrm{m_H}(s) \leq \mathrm{m_{ADM}}. \tag{18}$$

In the next few sections we will see how this constrains isometric embeddings of the manifolds into Euclidean space allowing us later to estimate the flat distance between these spaces and their limits.

2.2. **Riemannian Embedding into $\mathbb{E}^{m+1}$.** In this section we describe the Riemannian isometric embedding from our manifold M^m into $\mathbb{E}^{m+1}$ and basic consequences. Recall that a Riemannian isometric embedding is a diffeomorphism

$$\psi : M^m \to N^n \text{ such that } |\psi_* V| = |V| \;\; \forall V \in TM_p. \tag{19}$$

This is not an isometric embedding in the metric sense (see (66)).

Lemma 2.1. *Given $M^m \in \mathrm{RotSym}_m$, we can find a rotationally symmetric Riemannian isometric embedding of M^m into Euclidean space as the graph of some radial function $z = z(r)$ satisfying $z'(r) \geq 0$. In graphical coordinates, we have*

$$g = (1 + [z'(r)]^2)dr^2 + r^2 g_0, \tag{20}$$

with $r \geq r_{min}$ and the following formulae for scalar curvature, area, mean curvature, Hawking mass and its derivative in terms of the radial coordinate r:

$$\mathrm{R}(r) = \frac{m-1}{1+(z')^2}\left(\frac{z'}{r}\right)\left((m-2)\frac{z'}{r} + \frac{2z''}{1+(z')^2}\right) \tag{21}$$

$$A(r) = \omega_{m-1} r^{m-1} \tag{22}$$

$$H(r) = \frac{m-1}{r\sqrt{1+(z')^2}} \tag{23}$$

$$\mathrm{m_H}(r) = \frac{r^{m-2}}{2}\left(\frac{(z')^2}{1+(z')^2}\right) \tag{24}$$

$$\mathrm{m_H}'(r) = \frac{r^{m-1}}{2(m-1)}\mathrm{R} \tag{25}$$

This Riemannian isometric embedding is unique up to a choice of $z_{min} = z(r_{min})$.

Proof. First observe that by positivity of the Hawking mass, (13) and the lack of interior minimal surfaces (8), we have $f'(s) \in (0, 1)$. Set $r(s) = f(s)$ and observe that since s is a distance function,

$$s'(r) = \sqrt{1 + (z'(r))^2} \tag{26}$$

which is solvable because $s'(r) \geq 1$. We choose $z'(r) \geq 0$ which then determines $z(r)$ up to a constant z_{min}. The rest of the equations then follow from the corresponding equations in s. □

It is now easy to see the rotational symmetric case of the Positive Mass Theorem and Penrose Inequality which we restate here as the proof is important to the almost equality case:

Theorem 2.2. *Given $M^m \in \mathrm{RotSym}_m$ isometrically embedded into Euclidean space as above, we have*

$$\mathrm{m_H}(r_{min}) \leq \mathrm{m_H}(r) \leq \mathrm{m_{ADM}} \tag{27}$$

and if there is an equality then M^m is Euclidean space (when $\mathrm{m_{ADM}} = 0$) or a Riemannian Schwarzschild manifold of mass $\mathrm{m_{ADM}} > 0$,

$$g = \left(1 + \frac{2\mathrm{m_{ADM}}}{r^{m-2} - 2\mathrm{m_{ADM}}}\right) dr^2 + r^2 g_0. \tag{28}$$

Proof. The monotonicity of the Hawking mass follows from (15). When there is an equality we apply Lemma 2.1 to see that

$$\mathrm{m_{ADM}} = \frac{r^{m-2}}{2}\left(\frac{(z')^2}{1 + (z')^2}\right). \tag{29}$$

So

$$(1 + (z')^2) 2\mathrm{m_{ADM}} = r^{m-2}((z')^2) \tag{30}$$

and

$$2\mathrm{m_{ADM}} + (2\mathrm{m_{ADM}} - r^{m-2})((z')^2) = 0. \tag{31}$$

So

$$(z')^2 = \frac{-2\mathrm{m_{ADM}}}{(2\mathrm{m_{ADM}} - r^{m-2})}. \tag{32}$$

When $\mathrm{m_{ADM}} = 0$, $z'(r) = 0$ and $z = z_{min}$ is the Euclidean hyperplane.

Observe that r_{min} must then be 0 because $r_{min} > 0$ forces the existence of a minimal surface at the boundary, and $\partial B_0(r_{min})$ is not minimal in a hyperplane. □

The following lemma will be useful when examining the deep apparent horizons depicted in Figure 1 that may occur in sequences satisfying the hypothesis of Theorem 1.2:

Lemma 2.3. *When $r_{min} > 0$, we can replace the radial coordinate, r by the height coordinate, z, so that*

$$g = (1 + [r'(z)]^2) dz^2 + r(z)^2 g_0. \tag{33}$$

Then for $r \geq r_{disk}$ we have the following formulae for scalar curvature, area, mean curvature, Hawking mass and slope of the Hawking mass of a level in terms of the height coordinate z:

$$\mathrm{R}(z) = \frac{m-1}{r(1+(r')^2)}\left(\frac{m-2}{r} - \frac{2r''}{1+(r')^2}\right) = \frac{(m-1)((m-2)(1+(r')^2) - 2rr'')}{r^2(1+(r')^2)^2}$$

$$\mathrm{A}(z) = \omega_{m-1} r^{m-1} \tag{34}$$

$$\mathrm{H}(z) = \frac{(m-1)r'}{r\sqrt{1+(r')^2}} \tag{35}$$

$$\mathrm{m_H}(z) = \frac{r^{m-2}}{2(1+(r')^2)} \tag{36}$$

$$\mathrm{m}'_{\mathrm{H}}(z) = \frac{r^{m-1}r'}{2(m-1)}\mathrm{R}. \tag{37}$$

When $r_{min} = 0$, these formulas hold outside of a possibly Euclidean disk, $r^{-1}[r_{min}, r_{disk}]$ where $r_{disk} \in [r_{min}, \infty]$, When $r_{disk} = \infty$ we have Euclidean space.

Proof. By Lemma 2.1, $z'(r) \geq 0$. Let

$$r_{disk} = \sup\{r : z'(r) = 0\} \in [r_{min}, \infty]. \tag{38}$$

Then all the equations hold for $r > r_{disk}$.

By Lemma 2.1 we have $\mathrm{m_H}(r_{disk}) = 0$, so by the Positive Mass Theorem, $\mathrm{m_H}(r_{min}) = 0$, so $r_{min} = 0$ and $r^{-1}[0, r_{disk})$ is a ball. It is clearly a Euclidean disk by (20). □

2.3. **Bounding** $\mathrm{diam}(\partial M)$ **and** F'**.** In this subsection we use the ADM mass to provide Lipschitz control on $z = F(r)$ on annular regions [Lemma 2.5] and to bound the diameter of the boundary of the manifold [Lemma 2.4]. These lemmas will be applied later to prove our stability theorems [Theorem 1.2].

Lemma 2.4. *If $M^m \in$ RotSym then*

$$r_{min} \leq (2\mathrm{m_{ADM}})^{1/(m-2)}. \tag{39}$$

So $\mathrm{diam}(\partial M^m) \leq \pi\,(2\mathrm{m_{ADM}})^{1/(m-2)}$.

Proof. Assuming $r_{min} > 0$, we know by Lemma 2.3 with $z_{min} = z(r_{min})$ that

$$0 = \frac{(m-1)r'(z_{min})}{r_{min}\sqrt{1+(r'(z_{min})^2}} \tag{40}$$

because the boundary is a minimal surface. So $r'(z_{min}) = 0$ and the Hawking mass is

$$\mathrm{m_H}(z_{min}) = \frac{r_{min}^{m-2}}{2(1+0^2)}. \tag{41}$$

The lemma then follows from the monotonity of Hawking mass in (15). □

Lemma 2.5. *Using the graphical coordinates of Lemma 2.1, with $z = F(r)$, we have*

$$F'(r) \geq \sqrt{\frac{2m_1}{r^{m-2} - 2m_1}} \qquad \forall r \in [r_1, \infty) \tag{42}$$

for any $r_1 \geq r_{min}$ where $m_1 = \mathrm{m_H}(r_1)$ and

$$F'(r) \leq \sqrt{\frac{2\mathrm{m_{ADM}}}{r^{m-2} - 2\mathrm{m_{ADM}}}} \qquad \forall r \geq \max\left\{r_1, (2\mathrm{m_{ADM}})^{1/(m-2)}\right\} \tag{43}$$

where $\mathrm{m_{ADM}} = \mathrm{m_{ADM}}(M^m)$.

Proof. By the formulas in Lemma 2.1 and the monotonicity of Hawking mass in (15) we have the following for $r > r_1$:

$$\mathrm{m_H}(r_1) \leq \mathrm{m_H}(r) \leq \mathrm{m_{ADM}} \tag{44}$$

$$\mathrm{m_H}(r_1) \leq \frac{r^{m-2}}{2}\left(\frac{(z')^2}{1 + (z')^2}\right) \leq \mathrm{m_{ADM}} \tag{45}$$

$$2\mathrm{m_H}(r_1)(1 + (z')^2) \leq r^{m-2}(z')^2 \leq 2m(1 + (z')^2) \tag{46}$$

So we get

$$2\mathrm{m_H}(r_1) \leq (r^{m-2} - 2\mathrm{m_H}(r_1)(z')^2 \tag{47}$$

$$2\mathrm{m_{ADM}} \geq (r^{m-2} - 2\mathrm{m_{ADM}})(z')^2 \tag{48}$$

The first equation tells us that

$$z' \geq \sqrt{\frac{2m_1}{r^{m-2} - 2m_1}} \qquad \forall r > r_1. \tag{49}$$

The second implies that

$$z' \leq \sqrt{\frac{2\mathrm{m_{ADM}}}{r^{m-2} - 2\mathrm{m_{ADM}}}} \qquad \forall r \geq (2\mathrm{m_{ADM}})^{m-2}. \tag{50}$$

□

2.4. **Rotationally Symmetric Examples.** Here we review the key examples depicted in Figure 1 which inspired the use of the Intrinsic Flat Distance to estimate the stability of the Positive Mass Theorem. These are all well known examples but we present them for completeness of exposition.

Recall (15) implies that, in the rotationally symmetric setting, monotonicity of the Hawking mass on the symmetric spheres is *equivalent* to nonnegativity of scalar curvature. Therefore, we have the following lemma which will be useful for constructing examples:

Lemma 2.6. *There is a bijection between elements of* RotSym_m *and increasing functions* $\mathrm{m_H} : [r_{min}, \infty) \to \mathbb{R}$ *such that*

$$\mathrm{m_H}(r_{min}) = \frac{1}{2} r_{min}^{m-2} \tag{51}$$

and

$$\mathrm{m_H}(r) < \frac{1}{2} r^{m-2} \tag{52}$$

for $r > r_{min} \geq 0$. In this section we will call these functions admissible Hawking mass functions.

Proof. Given $M^m \in \mathrm{RotSym}$, apply Lemma 2.1, to determine $r_{min} \geq 0$. Since M^m has no closed interior minimal surfaces, $z'(r) > 0$ so

$$\mathrm{m_H}(r) = \frac{r^{m-2}}{2}\left(\frac{(z')^2}{1+(z'(r))^2}\right) < \frac{1}{2} r^{m-2}. \tag{53}$$

If $r_{min} = 0$, then $\mathrm{m_H}(r_{min}) = 0$. If $r_{min} > 0$, we have $\lim_{r\to r_{min}} z'(r) = \infty$, so we have (51).

Given an admissible Hawking function, $\mathrm{m_H} : [r_{min}, \infty) \to \mathbb{R}$, we define $z : [0, \infty) \to \mathbb{R}$, via the formula

$$z(\bar{r}) = \int_{r_{\min}}^{\bar{r}} \sqrt{\frac{2\mathrm{m_H}(r)}{r^{m-2} - 2\mathrm{m_H}(r)}}\, dr. \tag{54}$$

This determines a rotationally symmetric manifold. Since

$$z'(r) = \sqrt{\frac{2\mathrm{m_H}(r)}{r^{m-2} - 2\mathrm{m_H}(r)}} > 0 \qquad \forall r > r_{min} \tag{55}$$

we have no interior minimal surfaces. If $r_{min} = 0$ then $z'(r_{min}) = 0$ and if $r_{min} > 0$ then

$$\lim_{r\to r_{min}} z'(r) \geq \lim_{r\to r_{min}} \sqrt{\frac{2\mathrm{m_H}(r)}{r^{m-2} - 2\mathrm{m_H}(r)}} = \infty \tag{56}$$

so the boundary is an outermost minimal surface. □

We begin with the most basic example depicted in column two of that figure: Schwarzschild manifolds whose ADM mass converges to 0.

Example 2.7. *The Riemannian Schwarzschild space, M^m_{Sch} of mass* $\mathrm{m_{ADM}}$ *can be found by applying Lemma 2.6 with* $\mathrm{m_H}(r)$ *constant equal to* $\mathrm{m_{ADM}}$. *Its metric satisfies (28). These spaces are diffeomorphic to Euclidean space $\mathbb{E}^m$ with a ball of radius r_{min} removed. Fixing an area $\alpha_0 > 0$, we see that outside a rotationally symmetric sphere Σ_{α_0} of area* $\mathrm{Vol}_{m-1}(\Sigma_{\alpha_0}) = \alpha_0$ *the metric converges smoothly to the Euclidean metric. However, these manifolds are not diffeomorphic to Euclidean space and we do not have smooth convergence globally.*

Next we consider the deep gravity wells depicted in third and fourth columns of Figure 1. First we provide a general lemma describing which admissable Hawking masses lead to strongly vertical graphs $z = F(r)$:

Lemma 2.8. *Let $\epsilon > 0$. We choose an admissible Hawking mass function $\mathrm{m_H} : [r_{min}, \infty) \longrightarrow \mathbb{R}$ such that*

$$\mathrm{m_H}(r) \geq \frac{1}{2} r^{m-2}(1-\epsilon^2) \tag{57}$$

on the interval $[r_1, r_2]$. Then the distance from the level $r^{-1}(r_1)$ to the level $r^{-1}(r_2)$ in the corresponding manifold is greater than

$$z(r_2) - z(r_1) \geq (r_2 - r_1)\frac{(1-\epsilon^2)}{\epsilon^2}. \tag{58}$$

Proof. By (55) we have

$$z'(r) \geq \sqrt{\frac{r^{m-2}(1-\epsilon^2)}{r^{m-2} - r^{m-2}(1-\epsilon^2)}} = \frac{(1-\epsilon^2)}{\epsilon^2}. \tag{59}$$

□

We now apply this to state and prove the example of the deep horizon depicted in the third column of Figure 1:

Example 2.9. *Given $L > 0$ and $\alpha_0 > 0$ and $\delta > 0$ we claim we can construct $M^m \in \mathrm{RotSym}_m$ with $\mathrm{m_{ADM}}(M^m) < \delta$ such that the distance $d(\Sigma_{min}, \Sigma_{\alpha_0}) > L$ where Σ_{α_0} is a level set of area $\mathrm{Vol}_{m-1}(\Sigma_{\alpha_0}) = \alpha_0$ and Σ_{min} is either the boundary of M^m or the pole.*

In Section 5 we use this example to find a sequence of Riemannian manifolds whose ADM mass approaches 0 that does not converge in the Gromov-Hausdorff sense to Euclidean space due to the thin deep wells. [Example 5.3].

Proof. Let r_0 be defined so that $\omega_{m-1} r_0^{m-1} = \alpha_0$. Let

$$\mathrm{m_H}(r) = \delta' = \min\{\delta/2, r_0^{m-2}/2\} \text{ for } r \geq r_A. \tag{60}$$

Given any $\epsilon > 0$, choose $r_\epsilon \in (0, r_0)$ so that

$$\frac{1}{2} r_\epsilon^{m-2}(1-\epsilon^2) = \delta'. \tag{61}$$

If $\mathrm{m_H}(r)$ is a smooth function such that

$$\frac{1}{2} r^{m-2}(1-\epsilon^2) \leq \mathrm{m_H}(r) \leq \frac{1}{2} r^{m-2} \text{ for } r \in [r_\epsilon/2, r_\epsilon] \tag{62}$$

then

$$\begin{aligned}
z(r_\epsilon) - z(r_\epsilon/2) &\geq \frac{r_\epsilon(1-\epsilon^2)}{2\epsilon^2} && \text{by Lemma 2.8,} && (63)\\
&= \frac{(1-\epsilon^2)}{2\epsilon^2}\left(\frac{2\delta'}{(1-\epsilon^2)}\right)^{1/(m-2)} && \text{by (61),} && (64)\\
&\geq L && \text{for } \epsilon \text{ sufficiently small and fixed } \delta'. && (65)
\end{aligned}$$

Finally choose $r_{min} = r_\epsilon/4$ or 0 and choose a smooth admissible Hawking function satisfying (60) and apply Lemma 2.6. □

3. Isometric Embeddings and the Intrinsic Flat Distance

In this section we provide techniques for constructing explicit filling manifolds to estimate the Intrinsic Flat Distance.

3.1. **Review.**

In the definition of the Intrinsic Flat Distance, one uses metric isometric embeddings (a la Gromov):

$$\varphi : X \to Z \text{ such that } d_Z(\varphi(x_1), \varphi(x_2)) = d_X(x_1, x_2) \qquad \forall x_1, x_2 \in X. \tag{66}$$

In contrast, one often finds Riemannian isometric embeddings (a la Nash) as defined in (19). Riemannian isometric embeddings are not necessarily metric isometric embeddings.

The metric space property of a Riemannian isometric embedding is a length space property. Recall that a length space is a metric space (X, d) such that

$$d_X(x_1, x_2) = \inf\{L_X(C) : \ C(0) = x_1, \ C(1) = x_2\} \tag{67}$$

where the length, $L_X(C)$, of the curve $C : [0, 1] \to X$ is the rectifiable length using d_X. Given a rectifiably connected subset $Y \subset Z$, it has an induced metric

$$d_Y(y_1, y_2) = \inf\{L_Y(C) : \ C(0) = y_1, \ C(1) = y_2\} \tag{68}$$

where the length of the curve $C : [0, 1] \to Y \subset Z$ is the rectifiable length using d_Z. Observe that (Y, d_Y) (the induced metric) is then a length space while (Y, d_Z) (the restricted metric) is just a metric space and

$$d_Y(y_1, y_2) \geq d_Z(y_1, y_2). \tag{69}$$

Consider as an example

$$Y = \{(x, y, z) : \ x^2 + y^2 + z^2 = 1\} \subset \mathbb{E}^3. \tag{70}$$

Here the restricted metric is the distance measured using line segments while the induced metric or intrinsic metric is the distance measured in the sphere.

Riemannian manifolds are length spaces. If $\varphi : M \to N$ is a Riemannian isometric embedding, then $L_M(C) = L_N(\varphi \circ C)$ for all curves $C : [0, 1] \to M$. Thus φ is an isometric embedding from M to its image, $\varphi(M)$, where the image is endowed with the induced metric:

$$d_M(p_1, p_2) = d_{\varphi(M)}(\varphi(p_1), \varphi(p_2)) \geq d_N(\varphi(p_1), \varphi(p_2)) \qquad \forall p_1, p_2 \in M. \tag{71}$$

In fact it is an isometry onto its image with the induced length metric. However it is not an isometric embedding into N unless the image $\varphi(M)$ is convex in N. When $\varphi(M)$ is convex, the infimums are achieved by length minimizing curves that lie within the set, and so, in that case, it is an isometric embedding. A plane is convex in $\mathbb{E}^3$, so it is isometrically embedded. The equatorial sphere in a 3-sphere is isometrically embedded into the three sphere.

Many examples of isometric embeddings are given in [12] and in [30] as they are an essential ingredient towards the explicit computation of filling volumes and Intrinsic Flat Distances. Among these is the classic warped product:

Theorem 3.1. *Given a warped product manifold, $M^m = \mathbb{R} \times_f S^{m-1}$ with metric $g_M = dr^2 + f(r)^2 g_0$ where g_0 is the standard metric on S^{m-1}. This isometrically embeds into $N^{m+1} = \mathbb{R} \times_f S^m$ with metric $g_M = dr^2 + f(r)^2 g_0$ where g_0 is the standard metric on S^m via an isometric embedding which preserves the radial coordinate, r, and maps each sphere into the equatorial sphere of that level.*

Rotationally symmetric subsets of Euclidean space do not isometrically embed into Euclidean space. However, they can be viewed as warped products and be isometrically embedded into rotationally symmetric submanifolds of higher dimensional Euclidean space:

$$\varphi : M = \left\{(x, y, z) : z = f(x^2 + y^2)\right\} \to N = \left\{(x, y, z, w) : z = f(x^2 + y^2 + w^2)\right\}$$

where M and N are endowed with the induced length metrics and $\varphi(x, y, z) = (x, y, z, 0)$.

We close this review section with the following theorem which could be useful in applications of Intrinsic Flat Distance to general relativity. The isometric embeddings given in Nirenberg's theorem applied in the work of Shi-Tam satisfy the hypothesis of this theorem [24] [28].

Theorem 3.2. *If $\varphi : M^m \to \mathbb{E}^{m+1}$ is a Riemannian isometric embedding such that $\varphi(M^m) = \partial K$ where K is a closed convex set in E^{m+1}, then $\varphi : M^m \to Cl(\mathbb{E}^{m+1} \setminus K)$ is an isometric embedding.*

While this theorem must be classical, its proof isn't readily available for citation, so we include it here:

Proof. Let $p_1, p_2 \in M^m$ be joined by $C : [0, 1] \to Cl(\mathbb{E}^{m+1} \setminus K)$, which is the shortest among all such curves. We know the shortest exists by applying the Arzela Ascoli Theorem keeping in mind that a sequence of curves of decreasing length remains in a compact subset of $\mathbb{E}^3$.

If the image of C lies in ∂K, then we are done. Assume on the contrary, that is does not. Let

$$t_1 = \sup\{t : C([0, t]) \subset \partial K\} \in [0, 1) \tag{72}$$

and let

$$t_2 = \sup\{t : C((t_1, t)) \subset \mathbb{E}^{m+1} \setminus K\} \in (t_1, 1]. \tag{73}$$

Since C is a length minimizing curve, its restriction to $[t_1, t_2]$ is minimizing from $C(t_1)$ to $C(t_2)$. This segment lies in the open set $\mathbb{E}^{m+1} \setminus K$, so variation of arclength within this flat region proves it is a straight Euclidean line segment. Since $\varphi(C(t_1)), \varphi(C(t_2)) \subset K \subset \mathbb{E}^{m+1}$ and K is convex, this segment must lie in K. This is a contradiction. □

3.2. **Constructing Isometric Embeddings.** In this subsection we prove the following theorem which will later be applied to estimate the Intrinsic Flat Distances between spaces.

Theorem 3.3. *Let $\varphi : M \to N$ be a Riemannian isometric embedding and let*

$$C_M := \sup_{p,q\in M} \left(d_M(p,q) - d_N(\varphi(p),\varphi(q))\right). \tag{74}$$

Then if

$$Z = \{(x,0) : x \in N\} \cup \{(x,s) : x \in \varphi(M),\ s \in [0,S_M]\} \subset N \times [0,S_M] \tag{75}$$

where

$$S_M = \sqrt{C_M(\mathrm{diam}(M) + C_M)} \tag{76}$$

and $\psi : M \to Z$ is defined by $\psi(x) = (\varphi(x), S_M)$, is an metric isometric embedding into (Z, d_Z) where d_Z is the induced length metric from the isometric product metric on $N \times [0,S_M]$.

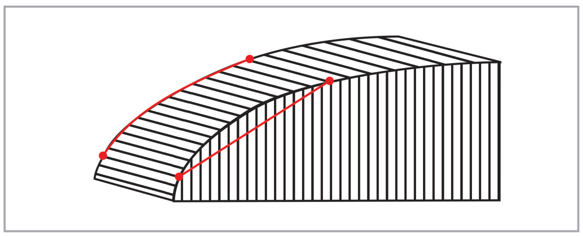

Figure 2. Explicit Isometric Embedding into Z

Later we will provide techniques for estimating the value of the "embedding constant" C_M.

Proof. First observe that $\psi : M \to N \times [0,S_M]$ is a Riemannian isometric embedding.

Let $p,q \subset M$. Let $C_i : [0,1] \to Z$ be curves parametrized proportional to arclength running from $\psi(p)$ to $\psi(q)$ such that

$$\lim_{i\to\infty} L(C_i) = d_Z(\psi(p),\psi(q)). \tag{77}$$

Observe that a closed ball in Z is compact, and the C_i are equicontinuous, so by the Arzela-Ascoli Theorem, a subsequence of the C_i converge to a length minimizing curve, $C : [0,1] \to Z$, parametrized proportional to arclength such that $C(0) = p$ and $C(1) = q$ and $L(C) = d_Z(\psi(p),\psi(q))$.

If the image of C lies in $\psi(M)$, then C is the shortest curve in $\psi(M) \subset Z$ from $\psi(p)$ to $\psi(q)$. Since $\psi : M \to N \times [0,S_M]$ is a Riemannian isometric embedding, $\psi : M \to \psi(M)$ is an isometry. Thus there is a curve $\gamma : [0,1] \to M$ running from p to q such that $\psi \circ \gamma = C$. Furthermore γ is length minimizing in M and parametrized proportional to arclength, so it is a minimizing geodesic in M. So

$$d_M(p,q) = L(\gamma) = L(C) = d_Z(\psi(p),\psi(q)). \tag{78}$$

Thus we need only show the image of C lies in $\psi(M)$.

We will write $C(t) = (x(t), s(t))$. Let

$$T_M = \{t \in [0,1] : C(t) \subset \varphi(M) \times [0,S_M].\} \tag{79}$$

and let

$$T_N = \{t \in [0,1] : C(t) \subset N \times \{0\}.\} \tag{80}$$

If $[t_1, t_2] \subset T_M$ then (since C is length minimizing in an isometric product) the length of this interval of C satisfies

$$L(C[t_1,t_2]) = \sqrt{d_{\varphi(M)}(x(t_2), x(t_1))^2 + (s(t_2) - s(t_1))^2} \tag{81}$$

with $x[t_1, t_2]$ the image of a minimizing geodesic segment in M and $s[t_1, t_2] = [s(t_1), s(t_2)]$. If $[t_1, t_2] \subset T_M$ and $t_i \subset T_N$, then $s(t_i) = 0$ so in fact this segment of C lies in $\varphi(M) \times \{0\}$. Thus

$$T_M = [0, m_1] \cup [m_2, 1]. \tag{82}$$

If $m_1 = m_2$, then we can apply (81) with $t_0 = 0$ and $t_1 = 1$ and the fact that $s(0) = s(1) = S_M$, to see that the image of C lies in $\varphi(M) \times \{0\} = \Psi(M)$ and we are done.

Assume on the contrary that $m_2 > m_1$. Observe that since the image of C lies in Z, $C : [m_1, m_2] \to N \times \{0\} \subset Z$. Furthermore

$$C(m_1), C(m_2) \in \varphi(M) \times \{0\}. \tag{83}$$

Since C is length minimizing in Z, it is length minimizing between $C(m_1) = (x(m_1), 0)$ and $C(m_2) = (x(m_2), 0)$. Thus

$$\begin{aligned} L(C[m_1, m_2]) &= d_{N\times\{0\}}(C(m_1), C(m_2)) && (84)\\ &= d_N(x(m_1), x(m_2)) && (85)\\ &\geq d_{\varphi(M)}(x(m_1), x(m_2)) - C_M. && (86) \end{aligned}$$

We will next sum up the lengths of the three segments of C to reach a contradiction. As this will involve the length on the isometric product regions, we first observe some properties on these regions. Let

$$X = X(a, b) = d_{\varphi(M)}(x(a), x(b)) \leq \mathrm{diam}(M). \tag{87}$$

By our choice of S_M we have

$$S_M^2 = C_M(\mathrm{diam}(M) + C_M) \geq XC_M + C_M^2. \tag{88}$$

and so

$$X^2 + S_M^2 > X^2 + 2XC_M/2 + C_M^2/4. \tag{89}$$

By (81), if $[a, b] \subset T_M$ and $|s(a) - s(b)| = S_M$ then

$$L(C[a, b]) = \sqrt{X(a, b)^2 + S_M^2} > X(a, b) + C_M/2. \tag{90}$$

Combining this with (86) and the fact that $s(0) = s(1) = S_M$ and $s(m_1) = s(m_2) = 0$, we have

$$L(C) = L(C[0, m_1]) + L(C[m_1, m_2]) + L(C[m_2, 1]) \tag{91}$$

where

$$L(C[0,m_1]) = \sqrt{X(0,m_1)^2+(s(0)-s(m_1))^2} \tag{92}$$
$$= \sqrt{X(0,m_1)^2+S_M^2} \tag{93}$$
$$> X(0,m_1)+C_M/2 \tag{94}$$
$$= d_{\varphi(M)}(x(0),x(m_1)) \tag{95}$$

$$L(C[m_1,m_2]) = X(m_1,m_2)-C_M) \tag{96}$$
$$= d_{\varphi(M)}(x(m_1),x(m_2)) \tag{97}$$

$$L(C[m_2,1]) = \sqrt{X(m_2,1)^2+(s(m_2)-s(1))^2} \tag{98}$$
$$= \sqrt{X(m_2,1)^2+S_M^2} \tag{99}$$
$$> X(m_1,1)+C_M/2 \tag{100}$$
$$= d_{\varphi(M)}(x(m_2),x(1)). \tag{101}$$

So that by the Triangle Inequality we have

$$L(C) > d_{\varphi(M)}(x(0),x(1)) = d_{\varphi(M)}(p,q). \tag{102}$$

This contradicts $L(C) = d_Z(\psi(p),\psi(q)) \le d_{\varphi(M)}(p,q)$. □

3.3. **Estimating the Intrinsic Flat Distance.** In this subsection we prove two general propositions that can be applied to bound the Intrinsic Flat Distance between Riemannian manifolds that have Riemannian isometric embeddings into a common Riemannian manifold. Recall the bound on the Intrinsic Flat Distance given in the introduction in (1) and (2) require a metric isometric embedding so we apply Theorem 3.3. The first proposition is clear and easy to see while the second is a bit more complicated but necessary to prove Theorem 1.2.

Proposition 3.4. *If $\varphi_i : M_i^m \to N^{m+1}$ are Riemannian isometric embeddings with embedding constants C_{M_i} as in (74), and if they are disjoint and lie in the boundary of a region $W \subset N$ then*

$$d_{\mathcal{F}}(M_1,M_2) \le S_{M_1}\left(\mathrm{Vol}_m(M_1)+\mathrm{Vol}_{m-1}(\partial M_1)\right) \tag{103}$$
$$+S_{M_2}\left(\mathrm{Vol}_m(M_2)+\mathrm{Vol}_{m-1}(\partial M_2)\right) \tag{104}$$
$$+\mathrm{Vol}_{m+1}(W)+\mathrm{Vol}_m(V) \tag{105}$$

where $V = \partial W \setminus (\varphi_1(M_1)\cup\varphi_2(M_2))$ where S_M are defined in (76).

Proof. We first create a piecewise smooth manifold,

$$Z^{m+1} = (M_1\times[0,S_{M_1}]) \ \cup\ (M_2\times[0,S_{M_2}]) \ \cup\ W^{m+1} \tag{106}$$

where the regions are glued together along the Riemannian isometric embeddings $\varphi_i(M_i) \subset B^{m+1}$ to $M_i\times\{0\}$ to form Z. Applying Theorem 3.3, we have metric isometric embeddings $\psi_i : M_i \to Z$ defined by $\psi_i(x) = (\varphi_i(x), S_M)$. Setting our filling manifold $B^{m+1} = Z^{m+1}$ as in (2), we then have an excess boundary

$$A^m = (\partial M_1)\times[0,S_{M_1}] \ \cup\ (\partial M_2)\times[0,S_{M_2}] \ \cup\ V. \tag{107}$$

The proposition then follows from (1). □

The next proposition will be applied to prove Theorem 1.2. It concerns pairs of manifolds which do not have global Riemannian isometric embeddings into a common manifold U^{m+1}:

Proposition 3.5. *If M_i^m are Riemannian manifolds and $U_i^m \subset M_i^m$ are submanifolds that have Riemannian isometric embeddings $\varphi_i : U_i^m \to N^{m+1}$ with embedding constants C_{U_i} as in (74), and if their images are disjoint and lie in the boundary of a region $B_0 \subset N$ then*

$$
\begin{aligned}
&(108) \qquad d_{\mathcal{F}}(M_1, M_2) \leq S_{U_1}\left(\mathrm{Vol}_m(U_1) + \mathrm{Vol}_{m-1}(\partial U_1)\right) \\
&(109) \qquad +S_{U_2}\left(\mathrm{Vol}_m(U_2) + \mathrm{Vol}_{m-1}(\partial U_2)\right) \\
&(110) \qquad + \mathrm{Vol}_{m+1}(B_1) + \mathrm{Vol}_m(V) \\
&(111) \qquad + \mathrm{Vol}_m(M_1 \setminus U_1) + \mathrm{Vol}_m(M_2 \setminus U_2)
\end{aligned}
$$

where $V = \partial B_1 \setminus (\varphi_1(U_1) \cup \varphi_2(U_2))$ where S_U are defined in (76).

Proof. Let $S_i = S_{U_i}$ as in Theorem 3.3. We first create a piecewise smooth manifold,

$$
\begin{aligned}
&(112) \qquad Z^{m+1} = (U_1 \times [0, 2S_1]) \cup (U_2 \times [0, 2S_2]) \cup B_1^{m+1} \\
&(113) \qquad \cup (M_1 \setminus U_1) \times [S_1, 2S_1] \cup (M_2 \setminus U_2) \times [S_2, 2S_2]
\end{aligned}
$$

where the regions are glued together along the Riemannian isometric embeddings $\varphi_i(U_i) \subset B_1^{m+1}$ to $U_i \times \{0\}$ and along $\partial U_i \times [S_i, 2S_i]$ to form Z. Applying Theorem 3.3, we have metric isometric embeddings $\psi_i : U_i \to Z$ defined by $\psi_i(x) = (\varphi_i(x), S_i)$. In fact these extend to isometric embeddings $\psi_i : M_i \to Z$ defined by $\psi_i(x) = (\varphi_i(x), S_i)$ since this is a metric isometric embedding on $M_i \setminus U_i$ and any path in Z running from $M_i \setminus U_i \times [S_i, 2S_i]$ to $U_i \times [0, 2S_i]$ must pass through $\partial U_i \times [S_i, 2S_i]$ and would be shorter if it stayed in $M_i \times \{S_i\}$.

Our filling manifold is chosen to be

$$
(114) \qquad B^{m+1} = (U_1 \times [0, S_1]) \cup (U_2 \times [0, S_2]) \cup B_1^{m+1}.
$$

Then by (2), we then have an excess boundary

$$
\begin{aligned}
&(115) \qquad A^m = (\partial U_1) \times [0, S_1] \cup (\partial U_2) \times [0, S_2] \cup V \\
&(116) \qquad \cup M_1 \setminus U_1 \cup M_2 \setminus U_2.
\end{aligned}
$$

The proposition then follows from (1). □

3.4. **The Embedding Constant for Graphs.** In this section we provide means for estimating the embedding constant, C_M, as defined in (74):

$$
(117) \qquad C_M := \sup_{p,q \in M} d_M(p, q) - d_N(\varphi(q), \varphi(q)).
$$

when the manifold M has a Riemannian isometric embedding $\varphi : M^m \to \mathbb{E}^n$

Theorem 3.6. *Let M^m be a compact Riemannian manifold with boundary defined by the graph*

$$M^m = \{(x,z) : z = F(x),\ x \in W\} \subset W \times \mathbb{R} \tag{118}$$

where $F : W \to \mathbb{R}$ is differentiable and W is a Riemannian manifold with boundary. Viewed as a Riemannian isometric embedding into $W \times \mathbb{R}$ the embedding constant satisfies

$$C_M \le 2\,\mathrm{diam}(W) \sup\{|\nabla F_x| : x \in W\}. \tag{119}$$

Later we will apply this theorem with

$$W = Ann_0(R_0, R_1) \subset \mathbb{E}^m \tag{120}$$

for rotationally symmetric F.

Remark 3.7. *If one examines the proof one can see that C_M is really bounded by an integral of $|\nabla F|$ over a length minimizing curve in W. However, the estimate we've written in Theorem 3.6 suffices for our purposes.*

Proof. Since M^m is compact, there exists a pair of points $p_0, p_1 \in M$ such that

$$C_M = d_{M^m}(p_0, p_1) - d_{W\times\mathbb{R}}(p_0, p_1). \tag{121}$$

We write $p_i = (x_i, z_i)$.

Let C be a length minimizing curve in $W \times \mathbb{R}$ from p_0 to p_1. We write $C(t) = (x(t), z(t)) \in W \times \mathbb{R}$. Then $x(t)$ is a length minimizing curve in W from x_0 to x_1 because it is the projection (in an isometric product) of a length minimizing curve. Let

$$h := d_W(x_0, x_1) \le \mathrm{diam}(W). \tag{122}$$

We now parametrize $C(t)$ and $x(t)$ so that $x : [0, h] \to W$ is parametrized by arclength and $x(0) = x_0$, $x(h) = x_1$, $z(0) = z_0$ and $z(h) = z_1$.

Observe that $\{(x(t), z) : t \in [0, h], z \in \mathbb{R}\}$ is isometric to a flat Euclidean strip $[0, h] \times \mathbb{R}$ with the metric restricted from $W \times \mathbb{R}$. The isometry $\psi(t, z) = (x(t), z)$. This implies that $z(t)$ is linear in t and

$$z'(t) = \frac{(z_1 - z_0)}{h} \tag{123}$$

Note that $x(t)$ is length minimizing in a manifold with boundary so it is not a smooth geodesic. However it is smooth away from a discrete set of points. Where it is smooth $g_W(x'(t), x'(t)) = 1$.

Define $\tilde{C}(t) = (\tilde{x}(t), \tilde{z}(t)) \in M \subset W \times \mathbb{R}$ where $\tilde{x}(t) = x(t)$ and $\tilde{z}(t) = F(x(t))$. Observe that where $x(t)$ is smooth, we have

$$|\tilde{z}'(t)| = |\nabla F_{x(t)}|. \tag{124}$$

Since $p_0, p_1 \in M$, $\tilde{z}(0) = z(0)$ and $\tilde{z}(h) = z(h)$. Thus $\tilde{C}$ is a curve from p_0 to p_1 in M.

Let γ be a length minimizing curve in M from p_0 to p_1. Then by (121)

$$C_M = L(\gamma) - L(C) \le L(\tilde{C}) - L(C). \tag{125}$$

Since $x(t)$ is smooth on a set of full measure in $[0, h]$, so are C and $\tilde{C}$, and we have

$$C_M = \int_0^h g_{W\times\mathbb{R}}(\tilde{C}'(t), \tilde{C}'(t))^{1/2}\, dt - \int_0^h g_{W\times\mathbb{R}}(C'(t), C'(t))^{1/2}\, dt \tag{126}$$

$$= \int_0^h (1 + (\tilde{z}'(t))^2)^{1/2} - (1 + (z'(t))^2)^{1/2}\, dt \tag{127}$$

$$= \int_0^h (1 + (\tilde{z}'(t))^2)^{1/2} - (1 + (z_1 - z_0)^2/h^2)^{1/2}\, dt. \tag{128}$$

Let

$$T = \{t \in [0, h] : |\tilde{z}'(t)| \geq |z_1 - z_0|/h\}. \tag{129}$$

Since $\tilde{z}$ is smooth away from a finite collection of points, we see that there exists

$$0 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n \leq h \tag{130}$$

such that

$$T = \bigcup_{i=1}^n [a_i, b_i]. \tag{131}$$

So

$$C_M \leq \sum_{i=1}^n \int_{a_i}^{b_i} (1 + (\tilde{z}'(t))^2)^{1/2} - (1 + (z_1 - z_0)^2/h^2)^{1/2}\, dt \tag{132}$$

$$= \sum_{i=1}^n \int_{a_i}^{b_i} \left(\int_{|z_1 - z_0|/h}^{|\tilde{z}'(t)|} \frac{d}{dy} \sqrt{y^2 + 1}\, dy \right) dt \tag{133}$$

$$= \sum_{i=1}^n \int_{a_i}^{b_i} \left(\int_{|z_1 - z_0|/h}^{|\tilde{z}'(t)|} \frac{y}{\sqrt{y^2 + 1}}\, dy \right) dt \tag{134}$$

$$\leq \sum_{i=1}^n \int_{a_i}^{b_i} \left(\int_{|z_1 - z_0|/h}^{|\tilde{z}'(t)|} 1\, dy \right) dt \tag{135}$$

$$= \sum_{i=1}^n \int_{a_i}^{b_i} |\tilde{z}'(t)| - |(z_1 - z_0)/h|\, dt \tag{136}$$

$$\leq \sum_{i=1}^n \int_{a_i}^{b_i} |\tilde{z}'(t)| + |z_1 - z_0|/h\, dt \tag{137}$$

$$\leq \int_0^h |\tilde{z}'(t)| + |z_1 - z_0|/h\, dt \tag{138}$$

$$= \int_0^h |\tilde{z}'(t)|\, dt + |z_1 - z_0| \tag{139}$$

Since $\tilde{z}(0) = z_0$ and $\tilde{z}(h) = z_1$ we have

$$C_M \leq \int_0^h |\tilde{z}'(t)|\, dt + \left| \int_0^h \tilde{z}'(t)\, dt \right| \tag{140}$$

$$\leq \int_0^h |\tilde{z}'(t)|\, dt + \int_0^h |\tilde{z}'(t)|\, dt \tag{141}$$

$$= 2 \int_0^h |\tilde{z}'(t)|\, dt \tag{142}$$

To obtain (119) we apply (124) and the fact that $h \leq \operatorname{diam}(W)$ from (122). □

Remark 3.8. *At the end of the proof we could have taken a much more subtle estimate of C_M as an integral of $|\nabla F|$ over a curve. However this overestimate suffices for our purposes.*

4. Positive Mass Stability Theorem

In this section we will prove Theorem 1.2 by constructing an explicit filling between the two tubular neighborhoods, $T_D(\Sigma_{\alpha_0}) \subset M^m$ and $T_D(\Sigma_{\alpha_0}) \subset Z^m$. We have a Riemannian embedding of M^m and $\mathbb{E}^m$ into $\mathbb{E}^{m+1}$ by Lemma 2.1 which we can use to fill in the space between the tubular neighborhood in M^m and its projection in $\mathbb{E}^m$. To create a metric isometric embedding we will attach a strip by applying Theorem 3.6 as in Figure 3.

To define the filling manifold and excess boundary more precisely, we recall that the radial function:

$$r(\Sigma_\alpha) = (\alpha/\omega_{m-1})^{m-1}. \tag{143}$$

Setting

$$r_{min} = \inf\{r(p) : r \in M^m\} \tag{144}$$

$$r_{D-} = \inf\{r(p) : p \in T_D(\Sigma_0) \subset M^m\} \tag{145}$$

$$r_0 = r(\Sigma_{\alpha_0}) = (\alpha_0/\omega_{m-1})^{m-1} \tag{146}$$

$$r_{D+} = \sup\{r(p) : p \in T_D(\Sigma_0) \subset M^m\}. \tag{147}$$

we see that $r_{min} \leq r_{D-} \leq r_{D+}$ all depend on the manifold while r_0 is an invariant for Theorem 1.2. Since

$$r_0 - D \leq r_{D-} \leq r_0 \leq r_{D+} \leq r_0 + D \text{ and } 0 \leq r_{min} \tag{148}$$

the tubular neighborhood in M^m projects to

$$r^{-1}(r_{D-}, r_{D+}) \subset T_D(\Sigma_0) \subset \mathbb{E}^m. \tag{149}$$

We will define a filling between the tubular neighborhood and this projection. See Figure 3. The region

$$A_0 = Ann_0(r_{D+}, r_0 + D) \subset T_D(\Sigma_{\alpha_0}) \subset \mathbb{E}^m, \tag{150}$$

will form part of our excess boundary and its volume will be estimated in Lemma 4.3. The inner region is more complicated as their may be a deep well in M^m.

To avoid difficulties with deep wells, in Lemma 4.1, we will choose $r'_\epsilon \in (0, r_0)$ where $r_0 = r(\Sigma_{\alpha_0}) = (\alpha_0/\omega_{m-1})^{1/(m-1)}$ and cut off the well

$$A_1 = r^{-1}(r_{D-}, r_\epsilon) \subset T_D(\Sigma_{\alpha_0}) \subset M^m \tag{151}$$

and the corresponding annulus

$$A_2 = A_{2,1} = Ann_0(r_0 - D, r_\epsilon) \subset T_D(\Sigma_{\alpha_0}) \subset \mathbb{E}^m \tag{152}$$

where

$$r_\epsilon = \max\{r'_\epsilon, r_{D-}\} \tag{153}$$

Note that when $r_0 - D \leq 0$, $A_2 = B_0(r_\epsilon)$ as depicted in Figure 3. The volumes of regions A_1 and A_2 are uniformly estimated in Lemma 4.1.

Note when $r'_\epsilon \leq r_{D-}$ our tubular neighborhood is not intersecting with a deep well and we have $A_1 = \emptyset$. In that case we set

$$A_2 = A_{2,2} = Ann_0(r_0 - D, r_{D-}) \subset T_D(\Sigma_{\alpha_0}) \subset \mathbb{E}^m. \tag{154}$$

The volume of A_2 is bounded uniformly in Lemma 4.4.

Next we choose Riemannian isometric embeddings of

$$r^{-1}(r_\epsilon, r_{D+}) = T_D(\Sigma_{\alpha_0}) \setminus A_1 \subset M^m \text{ and } r^{-1}(r_\epsilon, r_{D+}) = T_D(\Sigma_{\alpha_0}) \setminus (A_0 \cup A_2) \subset \mathbb{E}^m$$

into $r^{-1}(r_\epsilon, r_{D+}) \subset \mathbb{E}^{m+1}$ such that $\Sigma_{\alpha_\epsilon} \subset M^m$ and $\Sigma_{\alpha_\epsilon} \subset \mathbb{E}^m$ coincide. Lemma 2.1 determines this embedding up to a vertical shift, so this is possible.

This determines the region:

$$B_1 = \{(x_1, ...x_m, z) : \ z \in [0, F(r)],\ r \in (r_\epsilon, r_{D+})\} \subset r^{-1}(r_\epsilon, r_{D+}) \subset \mathbb{E}^{m+1}, \tag{155}$$

between these Riemannian isometric embeddings. The region B_1 is not a filling manifold. The Euclidean annulus has a metric isometric embedding, but not the region in M^m. So we add a strip

$$B_2 = [0, S_M] \times r^{-1}[r_\epsilon, r_{D+}] \subset [0, S_M] \times M \tag{156}$$

where width S_M is determined in Lemma 4.5 using Theorems 3.3 and 3.6 and isometrically embed

$$r^{-1}(r_\epsilon, r_{D+}) \subset M^m \text{ and } r^{-1}(r_\epsilon, r_{D+}) \subset \mathbb{E}^m \tag{157}$$

into $B_1 \cup B_2$ as in Figure 3.

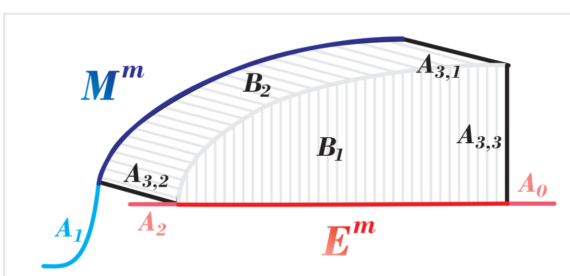

FIGURE 3. Explicit Isometric Embedding into Z

Applying Proposition 3.5, we then know

$$d_{\mathcal{F}}(\,T_D(\Sigma_{\alpha_0}) \subset M^m\,,\, T_D(\Sigma_{\alpha_0}) \subset \mathbb{E}^m\,) \le \mathrm{Vol}_{m+1}(B) + \mathrm{Vol}_m(A) \tag{158}$$

where $B = B_1 \cup B_2$ is the filling manifold and

$$A = A_0 + A_1 + A_2 + A_{3,1} + A_{3,2} + A_{3,3} \tag{159}$$

is the excess boundary with

$$A_{3,1} = [0, S_M] \times r^{-1}\{r_{D+}\} \subset [0, S_M] \times M, \tag{160}$$

$$A_{3,2} = [0, S_M] \times r^{-1}\{r_\epsilon\} \subset [0, S_M] \times M \tag{161}$$

$$A_{3,3} = r^{-1}\{r_{D+}\} \subset \partial B_1 \subset \mathbb{E}^{m+1}. \tag{162}$$

By estimating the volumes of these regions we will complete the proof of Theorem 1.2

4.1. **Cutting off the Deep Wells.** As seen in Figure 1, the manifolds with small mass can have arbitrarily deep wells. Here we determine where to cut them off.

Lemma 4.1. *Given $\epsilon > 0$, $D > 0$ $\alpha_0 > 0$ and $M^m \in$ RotSym, and a symmetric sphere $\Sigma_{\alpha_0} \in M^m$ of area α_0. Let*

$$\alpha_\epsilon = \min\left\{\epsilon/(16D), (\omega_{m-1}\epsilon/8)^{m/(m-1)}, \alpha_0\right\}. \tag{163}$$

Choose

$$r'_\epsilon = r(\Sigma_{\alpha_\epsilon}) = (\alpha_\epsilon/\omega_{m-1})^{1/(m-1)} > 0. \tag{164}$$

Then defining sets A_1 and $A_{2,1}$ in (151) and (152) respectively, we have

$$\mathrm{Vol}(A_1),\ \mathrm{Vol}(A_{2,1}),\ \mathrm{Vol}(B_0(r'_\epsilon) \subset \mathbb{E}^m) \le \epsilon/8. \tag{165}$$

Proof. Since $B_0(r_\epsilon) \subset \mathbb{E}^m$

$$\mathrm{Vol}_m(B_0(r'_\epsilon)) \le r'_\epsilon \alpha_\epsilon \le \alpha_\epsilon(\alpha_\epsilon/\omega_{m-1})^{1/(m-1)} < \epsilon/8. \tag{166}$$

Now $A_{2,1}$ and A_2 are empty unless $r_\epsilon = r'_\epsilon$ so we assume this for the rest of the proof. Then $A_{2,1} \subset B_0(r_\epsilon)$ has volume $< \epsilon/8$ as well.

Let $z_\epsilon = z(\Sigma_{\alpha_\epsilon})$ and $z_D = \min\{z(p) : p \in T_D(\Sigma_{\alpha_0}) \subset M^m\}$. Observe that $z_\epsilon - z_D < D$ because we chose $\alpha_\epsilon < \alpha_0$ and areas are monotone in RotSym and $r^{-1}(r_{D-}, r_{D+})$ is in a tubular neighborhood of radius D about Σ. Observe that the cylinder

$$C^m = \partial B_0(r_\epsilon) \times [z_D, z_\epsilon]. \tag{167}$$

has volume

$$\mathrm{Vol}_m(C^m) \le \alpha_\epsilon(z_\epsilon - z_D) \le \alpha_\epsilon D \le \epsilon/16. \tag{168}$$

Since $F'(z) \ge 0$, we can project the well, $A_1 \subset M^m$, radially outwards to C^m and vertically downwards to $B_0(r_\epsilon)$ to estimate the volume:

$$\mathrm{Vol}(A_1) \le \mathrm{Vol}_M(C^m) + \mathrm{Vol}_m(B_0(r_\epsilon)) < \epsilon/8. \tag{169}$$

□

4.2. **First Key Restrictions on δ for Theorem 1.2.** The first basic estimate follows immediately from Lemma 2.1, Lemma 2.5 and Lemma 2.4:

Lemma 4.2. *Given fixed $r_\epsilon > 0$, $D > 0$, $\alpha_0 > 0$, $m \in \mathbb{N}$, choose*

$$\delta < \delta(r_\epsilon) = (r_\epsilon/2)^{1/(m-2)}. \tag{170}$$

If $M^m \in \mathrm{RotSym}_m$ has $m_{ADM} < \delta$ then then M^m has a Riemannian isometric embedding into

$$\{z = F(r)\} \subset \mathbb{E}^{m+1} \tag{171}$$

where $F : [r_{min}, \infty) \to \mathbb{R}$ is an increasing function,

$$r_{min} \leq (2\delta)^{1/(m-2)} < r_\epsilon \tag{172}$$

and

$$|F'(r)| \leq Q(\delta, r_\epsilon) \qquad \forall r \geq r_\epsilon, \tag{173}$$

where

$$Q(\delta, r) := \sqrt{2\delta/(r_\epsilon^{m-2} - 2\delta)}. \tag{174}$$

Observe that

$$\lim_{\delta \to 0} Q(\delta, r_\epsilon) = 0 \tag{175}$$

for fixed r_ϵ.

Proof. Lemma 2.1 provides the Riemannian isometric embedding and Lemma 2.4 provides (172). Lemma 2.5 and the fact that $m_{ADM}(M) < 2\delta$ then implies that

$$|F'(r)| \leq Q(\delta, r) := \sqrt{2\delta/(r^{m-2} - 2\delta)} \qquad \forall r \geq (2\delta)^{1/(m-2)}. \tag{176}$$

By our choice of δ, $r_\epsilon > (2\delta)^{m-2}$, so we get (173) by applying the fact that $Q(\delta, r)$ decreases in r. □

Since we have already controlled all regions with $r < r_\epsilon$ in the last subsection, we can control the rest of the regions by taking δ small enough that we can apply Lemma 4.2.

4.3. **From the Projected Set to Tubular Neighborhood.** Here we estimate the volumes of the regions between the projected set $r^{-1}(r_{D-}, r_{D+}) \subset \mathbb{E}^m$ and the tubular neighborhood $T_D(\Sigma_{\alpha_0}) \subset \mathbb{E}^m$ proving Lemma 4.3 and Lemma 4.4.

Lemma 4.3. *Given $D > 0$, $\alpha_0 > 0$, $m \in \mathbb{N}$, Choosing $\delta > 0$ such that*

$$(2\delta)^{m-2} < r_0/2. \tag{177}$$

If $M^m \in \mathrm{RotSym}_m$ has $m_{ADM} < \delta$ then

$$\mathrm{Vol}_m(A_0) \leq DQ(\delta, r_0)\omega_{m-1}(r_0 + D)^{m-1} \tag{178}$$

Proof. By our choice of δ we know that

$$|F'(r)| \le Q(\delta, r_0) \qquad \forall r \ge r_0. \tag{179}$$

By the formula for arclength

$$\begin{aligned} r_0 + D - r_{D+} &= r_0 - r_{D+} + \int_{r_0}^{r_{D+}} \sqrt{1 + F'(r)^2}\, dr && (180)\\ &\le r_0 - r_{D+} + (r_{D+} - r_0)(1 + Q(\delta, r_0)) && (181)\\ &\le (r_{D+} - r_0) Q(\delta, r_0) \le D Q(\delta, r_0) && (182)\\ & && (183) \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{Vol}(A_0) &\le \mathrm{Vol}(r^{-1}(r_{D+}, r_0 + D) \subset \mathbb{E}^m) && (184)\\ &\le (r_0 + D - r_{D+})\, \omega_{m-1}(r_0 + D)^{m-1} && (185)\\ &\le D Q(\delta, r_0) \omega_{m-1}(r_0 + D)^{m-1} && (186) \end{aligned}$$

and the lemma follows. □

Recall that region $A_{2,2}$ defined in (154) is only defined when $r_\epsilon \le r_{D-}$. So the lemma estimating it's volume assumed this condition:

Lemma 4.4. *Given $r_\epsilon > 0$ $D > 0$, $\alpha_0 > 0$, $m \in \mathbb{N}$, we choose δ is in Lemma 4.2. If $M^m \in \mathrm{RotSym}_m$ has $m_{ADM} < \delta$ the region $A_{2,2}$ defined in (154) satisfies*

$$\mathrm{Vol}_m(A_{2,2}) \le D Q(\delta, r_\epsilon) \omega_{m-1}(r_0)^{m-1}. \tag{187}$$

Proof. By our choice of δ we know that

$$|F'(r)| \le Q(\delta, r_\epsilon) \qquad \forall r \ge r_{D-} \ge r_\epsilon. \tag{188}$$

By the formula for arclength

$$\begin{aligned} r_{D-} - (r_0 - D) &= r_{D-} - r_0 + \int_{r_{D-}}^{r_0} \sqrt{1 + F'(r)^2}\, dr && (189)\\ &\le r_{D-} - r_0 + (r_0 - r_{D-})(1 + Q(\delta, r_\epsilon)) && (190)\\ &\le (r_0 - r_{D-}) Q(\delta, r_0) \le D Q(\delta, r_\epsilon) && (191)\\ & && (192) \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{Vol}(A_{2,2}) &= \mathrm{Vol}(r^{-1}(r_0 - D, r_{D-}) \subset \mathbb{E}^m) && (193)\\ &\le D Q(\delta, r_\epsilon) \omega_{m-1}(r_0)^{m-1} && (194) \end{aligned}$$

and the lemma follows. □

4.4. **Choosing the Width of the Strip.**

Lemma 4.5. *Given $r_\epsilon > 0$ $D > 0$, $\alpha_0 > 0$, $m \in \mathbb{N}$, we choose δ is in Lemma 4.2. If $M^m \in$ RotSym and $\mathrm{m}_{\mathrm{ADM}}(M^m) < \delta$, then the region $r^{-1}(r_\epsilon, r_{D+})$ isometrically embeds into the filling manifold $B_1 \cup B_2$ of (155) and (156) where*

$$S_M = S(\delta, r_\epsilon, D, r_0) = \sqrt{C(2D + \pi r_0 + C)} \tag{195}$$

with

$$C = C(D, r_0, \delta, r_\epsilon) = (4D + 2\pi r_0)Q(\delta, r_\epsilon) \tag{196}$$

Proof. We begin by applying Theorem 3.3, to the Riemannian isometric embedding of $r^{-1}(r_\epsilon, r_{D+}) \subset M^m$ into $W \times \mathbb{R} \subset \mathbb{E}^{m+1}$ where $W = Ann_0(r_\epsilon, r_{D+}) \subset \mathbb{E}^m$.

Since $r^{-1}(r_\epsilon, r_{D+}) \subset T_D(\Sigma_{\alpha_0})$, we have

$$\mathrm{diam}(W) \le \mathrm{diam}(r^{-1}(r_\epsilon, r_{D+})) \le 2D + \mathrm{diam}(\Sigma_{\alpha_0}) = 2D + \pi r_0. \tag{197}$$

By Lemma 4.2 have a bound on F'. giving us the embedding constant $C_M \le C$ given above and by Theorem 3.6, the strip must have width S_M given above. □

4.5. **Volume Estimates and the Proof of Theorem 1.2.** The proof of Theorem 1.2 is completed by estimating the volumes of the regions depicted in Figure 3 and applying (158):

Proof. Given any $\epsilon > 0$, $D > 0$, $\alpha_0 > 0$, $m \in \mathbb{N}$ we choose

$$r'_\epsilon > 0 \tag{198}$$

depending only on ϵ and α_0 exactly as in Lemma 4.1. We set $r_0 > 0$ such that $\alpha_0 = \omega_{m-1} r_0^{m-1}$.

We choose

$$\delta < \delta(r_\epsilon) \tag{199}$$

as in Lemma 4.2. We will refine it further later in (199), (202), (207), (211), (214) and (217) to obtain $\delta = \delta(\epsilon, D, \alpha_0, m) > 0$.

Assume $M^m \in \mathrm{RotSym}_m$ has ADM mass $\mathrm{m}_{\mathrm{ADM}}(M) < \delta$.

We set

$$r_\epsilon = \max\{r'_\epsilon, r_{D-}\}. \tag{200}$$

as in (153). When $r_\epsilon \ge r_{D-}$ we apply Lemma 4.1, (**??**) and (**??**) to see that

$$\mathrm{Vol}_m(A_1) + \mathrm{Vol}_m(A_2) \le \epsilon/8 + \epsilon/8 = \epsilon/4. \tag{201}$$

When $r_\epsilon < r_{D-}$, then $A_1 = \emptyset$ and by Lemma 4.4, we obtain the same estimate as long as we choose $\delta > 0$ is chosen small enough that:

$$DQ(\delta, r_\epsilon)\omega_{m-1}(r_0)^{m-1} < \epsilon/8. \tag{202}$$

This second restriction on δ also suffices to obtain

$$\mathrm{Vol}_m(A_0) < \epsilon/8. \tag{203}$$

By Lemma 2.1 we have a Riemannian isometric embedding of $r^{-1}(r_\epsilon, r_{D+})$ into $\{z = F(r)\} \subset \mathbb{E}^{m+1}$ and may define B_1 as in (155). We then have

$$\mathrm{Vol}_{m+1}(B_1) = \int_{r_\epsilon}^{r_{D+}} (F(r) - F(r_\epsilon))\omega_{m-1} r^{m-1}\, dr \tag{204}$$

$$\leq (r_{D+} - r_\epsilon)\omega_{m-1} r_{D+}^{m-1}(F(r_{D+}) - F(r_\epsilon)) \tag{205}$$

$$\leq 2D\omega_{m-1}(r_0 + D)^{m-1} \int_{r_\epsilon}^{r_{D+}} F'(r)\, dr < \epsilon/8. \tag{206}$$

as long as δ is chosen small enough that

$$4D^2\omega_{m-1}(r_0 + D)^{m-1} Q(r_\epsilon, \delta) < \epsilon/8. \tag{207}$$

Applying Lemma 4.5 to create region B_2 as in (156) such that

$$\mathrm{Vol}_{m+1}(B_2) = S_M \,\mathrm{Vol}(r^{-1}(r_\epsilon, r_{D+}) \subset M^m) \tag{208}$$

$$= S_M \int_{r_\epsilon}^{r_{D+}} \sqrt{1 + F'(r)^2}\, \omega_{m-1} r^{m-1}\, dr \tag{209}$$

$$= S_M \int_{r_\epsilon}^{r_{D+}} (1 + F'(r))\, \omega_{m-1} r^{m-1}\, dr < \frac{\epsilon}{8} \tag{210}$$

as long as δ is chosen small enough that

$$S(\delta, r_\epsilon, D, r_0) 2D\omega_{m-1}(r_0 + D)^{m-1} Q(\delta, r_\epsilon) < \epsilon/8. \tag{211}$$

By (160), (161) we have

$$\mathrm{Vol}_m(A_{3,1}) = S_M\omega_{m-1} r_{D+}^{m-1} \leq S_M\omega_{m-1}(r_0 + D)^{m-1} < \epsilon/12 \tag{212}$$

$$\mathrm{Vol}_m(A_{3,2}) = S_M\omega_{m-1} r_\epsilon^{m-1} \leq S_M\omega_{m-1}(r_0)^{m-1} < \epsilon/12 \tag{213}$$

as long as δ is chosen small enough that

$$S(\delta, r_\epsilon, D, r_0)\omega_{m-1}(r_0 + D)^{m-1} < \epsilon/12 \tag{214}$$

By (162) we have

$$\mathrm{Vol}_m(A_{3,3}) = \omega_{m-1} r_{D+}^{m-1}(F(r_{D+}) - F(r_\epsilon)) \tag{215}$$

$$\leq \omega_{m-1}(r_0 + D)^{m-1} Q(\delta, r_\epsilon) < \epsilon/12 \tag{216}$$

as long as δ is chosen small enough that

$$\omega_{m-1}(r_0 + D)^{m-1} Q(\delta, r_\epsilon) < \epsilon/12. \tag{217}$$

The theorem follows from (158) summing over all these volumes. □

Remark 4.6. *Note that we have linear scaling on*

$$m(A)^{1/m} + m(B)^{1/m+1} \tag{218}$$

in this proof. Redefining the Intrinsic Flat Distance in this way might be worth investigating as it is apparently still a distance. Such a redefinition appears to induce the same intrinsic flat topology on the space of Riemannian manifolds. Recall that the flat distance was originally defined by Federer-Fleming [7] *to be a norm: linear in multiplication of an integral current by a magnitude. This property should only be abandoned with caution. See Remark 6.9.*

5. Gromov-Hausdorff Distance

In this section we review the Gromov-Hausdorff distance between Riemannian manifolds, provide new estimates for estimating the Gromov-Hausdorf distance [Propositions 5.1 and 5.2] based on the embedding constants defined in Theorem 3.3 and the prove in Example 5.3 that the Positiv Mass Theorem is not stable with respect to the Gromov-Hausdorff distance.

Recall that the Gromov-Hausdorff distance was first defined by Gromov in [14] as follows:

$$d_(M_1^m, M_2^m) = \inf_{\varphi_i : M_i \to Z} d_H^Z\left(\varphi_1(M_1), \varphi_2(M_2)\right), \tag{219}$$

where the infimum is taken over all metric spaces Z and all metric isometric embeddings $\varphi_i : M_i \to Z$ and where the Hausdorff distance in Z between two subsets X_1 and X_2 is

$$d_H(X_1, X_2) = \inf\{\rho > 0 : \; X_1 \subset T_\rho(X_2) \text{ and } X_2 \subset T_\rho(X_1). \tag{220}$$

5.1. New Estimates using Embedding Constants.

Naturally the techniques given in Section 3 may also be applied to estimate the Gromov-Hausdorff distance. In particular we see that Theorem 3.4 implies the following proposition much as it implies Proposition 3.4:

Proposition 5.1. *If M_i^m are Riemannian manifolds have Riemannian isometric embeddings $\varphi_i : M_i^m \to N^{m+1}$ with embedding constants C_{M_i} as in (74), and if their images are disjoint and lie in the boundary of a region $B_0 \subset N$ then*

$$d_{GH}(M_1, M_2) \quad \leq \quad S_{M_1} + S_{M_2} + d_H^N(\varphi_1(M_1), \varphi_2(M_2)) \tag{221}$$

where $S_{M_i} = \sqrt{C_{M_1}(\mathrm{diam}(M_i) + C_{M_i})}$.

Notice how the Gromov-Hausdorff distance does not allow one to cut off a well using only its volume to estimate it. The depth of the well will contribute to the distance. In place of Proposition 3.5 we have:

Proposition 5.2. *If M_i^m are Riemannian manifolds and $U_i^m \subset M_i^m$ are submanifolds that have Riemannian isometric embeddings $\varphi_i : U_i^m \to N^{m+1}$ with embedding constants C_{U_i} as in (74), and if their images are disjoint and lie in the boundary of a region $B_0 \subset N$ then*

$$d_{GH}(M_1, M_2) \quad \leq \quad S_{U_1} + S_{U_2} + d_H^N(\varphi_1(U_1), \varphi_2(U_2)) \tag{222}$$

$$+ \sup_{x \in M_1 \setminus U_1} d_{U_1}(x, M_1) + \sup_{x \in M_2 \setminus U_2} d_{U_2}(x, M_2) \tag{223}$$

where $S_{M_i} = \sqrt{C_{M_1}(\mathrm{diam}(M_i) + C_{M_i})}$.

If the regions $M \setminus U$ is a deep well, then Proposition 5.1 will provide a very poor estimate for the Gromov-Hausdorff distance between the spaces. In fact the spaces need not be close at all.

5.2. **Wells of Arbitrary Depth.** The Positive Mass Theorem is not stable with respect to the Gromov-Hausdorff distance. In fact, the manifolds given in Example 2.9 are close to Euclidean space with a line segment attached to it, $\mathbb{E}^m \cup [0, L]$ where the line segment can have arbitrary length. That is:

Example 5.3. *Given any $L_0 > 0$ there exists a sequence $M_j^m \in \mathrm{RotSym}_m$ such that $\lim_{j\to\infty} m_{\mathrm{ADM}}(M_j^m) = 0$ and the pointed Gromov-Hausdorff limit of M_j^m is $\mathbb{E}^m \cup [0, L]$ in the sense that for any fixed $\alpha) > 0$ and any $D > 0$ there exists $D_j \to D$ such that*

$$d_{GH}(B_{p_j}(D_j) \subset M^m, B_0(D) \subset \mathbb{E}^m \cup [0, L]) < \epsilon \tag{224}$$

where $p_j \in \Sigma_{\alpha_0} \subset M_j$.

Proof. Let $r_0 = (\alpha_0/\omega_m)^{1/(m-1)}$. Take $\delta_j = 1/j$ and and take M_j to be the manifold in Example 2.9 with

$$L = d_M(\Sigma_{\alpha_0}, \Sigma_{min}) = L_0 + r_0. \tag{225}$$

and $m_{ADM}(M_j) < \delta_j$.

Now fix $D > 0$. We isometrically embed the tubular neighborhoods $T_D(\Sigma_{\alpha_0}) \subset M_j$ and $T_D(\Sigma_{\alpha_o}) \subset \mathbb{E}^m$ into Z as in the proof of Theorem 1.2. We attach $\{0\} \in [0, L_0]$ to Euclidean space at a point in Σ_{r_ϵ} and then the interval runs down the well in M_j to the bottom of the well at $L_0 + r_0 - r_\epsilon$ and a little further a distance r_ϵ as an extra segment. We extend Z as well.

Thus $M_j \subset T_{\rho_j}(\mathbb{E}^m \cup [0, L_0])$ where

$$\rho_j = \max\{F(r_D) - F(r_\epsilon) + S_{M_j}, \pi r_\epsilon\}. \tag{226}$$

On the other hand $\mathbb{E}^m \cup [0, L_0] \subset T_{\rho'_j}(M_j)$ where

$$\rho'_j = \max\{r_\epsilon, F(r_D) - F(r_\epsilon) + S_{M_j}\} \tag{227}$$

since $d_{\mathbb{E}^m}(0, \partial B_p(r_\epsilon)) = r_\epsilon$ and the segment has extra length r_ϵ. Thus

$$d_{GH}\left(T_D(\Sigma_{\alpha_0}) \subset M_j^m, T_D(\Sigma_{\alpha_0}) \subset \mathbb{E}^m \cup [0, L_0]\right) \tag{228}$$

$$\leq \; d_H^Z\left(T_D(\Sigma_{\alpha_0}) \subset M_j^m, T_D(\Sigma_{\alpha_0}) \subset \mathbb{E}^m \cup [0, L_0]\right) \;<\; \max\{\rho_j, \rho'_j\}. \tag{229}$$

As $\delta_j \to 0$, we have $r_\epsilon \to 0$ and $S_{M_j} \to 0$ in the proof of Theorem 1.2 so $\rho_j, \rho'_j \to 0$.

Since this is true for all D, we can exhaust the space with the tubular neighborhoods and obtain the pointed Gromov-Hausdorff onvergence. □

5.3. **Arbitrarily Dense Collections of Wells.** If we remove the requirement that a manifold be rotationally symmetric then we can introduce more than one well. In fact we can create asymptotically flat manifolds, M_j^m with positive scalar curvature and $M_{ADM}(M_j) \to 0$ that have increasingly dense collections of wells [Example 5.6]. Such a sequence of manifolds doesn't even have a subsequence converging in the Gromov-Hausdorff sense.

These examples are based upon the work of Schoen-Yau and Gromov-Lawson [25][13] who have proven that if one has a manifold of constant sectional curvature, then one can attach a well of arbitrary depth and thinness to that manifold

maintaining positive scalar curvature. For this reason we are limiting ourselves to dimension three in the construction of the example, however, similar examples should exist in higher dimensions as well.

We begin by creating an element of RotSym_m with stripes of positive sectional curvature.

Recall that by Lemma 2.6 we need only create an admissible Hawking function with the desired properties to produce an element of RotSym_m.

Lemma 5.4. *A manifold $M^m \in \mathrm{RotSym}$ has constant sectional curvature, $K > 0$, on $r^{-1}(a, b)$ iff $r^{-1}(a, b)$ is an annulus in a sphere of radius $1/K^{1/2}$ iff $\mathrm{m_H}(r) = r^m K/2$ for $r \in (a, b)$.*

Proof. If it is an annulus in a sphere, then $(z-\zeta)^2 + r^2 = 1/K$, so $2(z-\zeta)z' + 2r = 0$ and thus $z' = -r/(z-\zeta)$ and by Lemma 2.1

$$\mathrm{m_H}(r) = \frac{r^{m-2}}{2} \frac{r^2/(z-\zeta)^2}{1 + r^2/(z-\zeta)^2} = \frac{r^m}{2(r^2 + (z-\zeta)^2)} = \frac{r^m K}{2}. \tag{230}$$

On the other hand, if $\mathrm{m_H}(r) = r^3 K/2$, then (54) defines a function $z(r)$ uniquely up to a constant. Since $z(r) = \sqrt{(1/K - r^2} + \zeta$ satisfies the equation, we must lie on a sphere. □

Example 5.5. *Fix $\delta > 0$. Given any increasing sequence,*

$$\{r_1, r_2, ...\} \subset [\mathrm{m_{fix}}/2, \infty), \tag{231}$$

there exists $M^3 \in \mathrm{RotSym}_3$ with constant sectional curvature on stripes $r^{-1}(a_j, b_j)$ where $(a_j, b_j) \subset [r_{2j-1}, r_{2j}]$ and $\mathrm{m_{ADM}}(M) < \delta$ and $\partial M = \emptyset$.

Proof. Recall that an admissable Hawking function need only be increasing and satisfy

$$\mathrm{m_H}(r) \le h(r) := \min\{r/2, \mathrm{m_{ADM}}\}. \tag{232}$$

For each j, choose the sectional curvature for the j^{th} annulus to be K_j satisfying $r_{2j}^3 K_j/2 = h(r_{2j})$. Observe that K_j is a decreasing sequence and

$$r^3 K_{j+1}/2 < r^3 K_j/2 < h(r) \text{ for } r < r_{2j}. \tag{233}$$

We now define $a_j < b_j$ inductively. Let $a_1 = r_1$. So $a_1^3 K_1/2 < h(a_1)$.

Next choose $b_j \in (a_j, (a_j + r_{2j})/2)$ satisfying

$$b_j^3 K_j/2 \le (a_j^3 K_1/2 + h(b_j))/2 < h(b_j). \tag{234}$$

Finally choose $a_{j+1} \in (r_{2j+1}, r_{2j+2})$ satisfying

$$a_{j+1}^3 K_{j+1}/2 \in (b_j^3 K_j/2, h(a_{j+1})) \tag{235}$$

which exists by our choice of K_j. We can choose any smooth increasing function $\mathrm{m_H} : [0, \infty) \to [0, \delta)$ such that

$$\mathrm{m_H}(r) = r^3 K_j/2 \text{ for } r \in [a_j, b_j] \tag{236}$$

and then apply Lemma 5.4 and 2.6. □

Example 5.6. *There exists a sequence of asymptotically flat manifolds M_i^3 with no interior minimal surfaces and empty boundary and $\lim_{i\to\infty} \mathrm{m}_{\mathrm{ADM}}(M_i) = 0$ such that for any $\alpha_0, D > 0$ the sequence of regions $T_D(\Sigma) \subset M_i$ where $\mathrm{Vol}_2(\Sigma) = \alpha_0$ converge in the intrinsic flat sense to $T_D(\Sigma) \subset \mathbb{E}^m$ but do not even have Lipschitz or Gromov-Hausdorff converging subsequences.*

Recall that Gromov's Compactness Theorem states that a sequence of compact metric spaces X_j has a subsequence converging to a compact metric space X if and only if there is a uniform bound on the number of disjoint balls of any given radius in the space [14]. In particular, a sequence of pointed Riemannian manifolds (M_j, p_j) has no subsequence converging in the pointed Gromov-Hausdorff sense if there is no uniform bound on the number $N(r, R)$ of disjoint balls of radius r lying in $B_{p_j}(R)$. Here we will construct such a sequence of M_j by gluing in increasingly many thin deep wells each of which contains a ball of radius r.

Proof. Fix $i \in \mathbb{N}$, $\delta = 1/i$, and choose a sequence $r_j = j/i$. Then by Lemma 5.4, there exists $\bar{M}^3 \subset \mathrm{RotSym}_3$ with $\mathrm{m}_{\mathrm{ADM}}(M^3) = 1/i$ that has stripes of constant sectional curvature on annular regions

$$r^{-1}(a_j, b_j) \subset r^{-1}[(2j-1)/i, 2j/i]. \tag{237}$$

By Schoen-Yau and Gromov-Lawson [25] [13], we can remove arbitrarily small balls, $B_{q_j}(\rho_j) \subset r^{-1}(a_j, b_j)$ for $j = 1$ to $(2i)^2$ and attach arbitrarily thin and deep wells, W_j, to each of these annular regions while maintaining nonnegative scalar curvature and without changing the metric outside the removed balls. In particular we can ensure that all the attached wells, W_j, have a depth

$$\max\{d(x, \partial W_j) : x \in W_j\} = 2D \tag{238}$$

and we can ensure that

$$\sum_{j=1}^{(2i)^2} \Big(\mathrm{Vol}_m(W_j) + \mathrm{Vol}_{m=1}(W_j) + \mathrm{Vol}_m(B_{q_j}(\rho_j)) + \mathrm{Vol}_m(\partial B_{q_j}(\rho_j))\Big) < 1/i. \tag{239}$$

and $\mathrm{diam}(\partial B_{q_j}(\rho_j)) < d_i$. We can also require that all wells satisfy

$$W_j \subset T_D(\Sigma) \setminus \Big(T_{D/10}(\Sigma) \cup T_{D/10}(\partial \bar{M}_j)\Big). \tag{240}$$

This gives us a non-rotationally symmetric manifold M^3 which is asymptotically flat with $\mathrm{m}_{\mathrm{ADM}}(M) = \delta$ such that for $\Sigma = r^{-1}(s_0)$ where s_0 is rational we have

$$d_{\mathcal{F}}(T_D(\Sigma) \subset M^3, T_D(\Sigma) \subset \bar{M}^3) < 1/i + \sqrt{C_i(C_i + 2D + \pi r(\Sigma))} \tag{241}$$

where C_i is the embedding constant of

$$\varphi_i : M^3 \setminus \bigcup W_j \to \bar{M}^3. \tag{242}$$

We may choose d_i sufficiently small to guarantee $\lim_{i\to\infty} C_i \to 0$. Applying Theorem 1.2 to $\bar{M}^3$ we have the claimed intrinsic flat convergence.

On the other hand for fixed s_0, and increasing i we have increasingly many wells contained in$T_D(\Sigma) \subset M_i$. Since each well has depth $2D$, the boundary

of$T_D(\Sigma) \subset M_i$ has increasingly many components. So clearly we do not have Lipschitz convergence to$T_D(\Sigma) \subset m_{\text{Sch}}$ even if we take a subsequence.

Also observe that if $\partial W_j \subset T_{D/3}(\Sigma) \subset M_i$ then

$$d_{M_i}\left(W_j \cap \partial T_D(\Sigma), \partial W_j\right) > D/3. \tag{243}$$

Thus balls of radius $D/3$ about $p_j \in W_j \cap \partial T_(D/3)(\Sigma)$ are pairwise disjoint and contained in $T_D(\Sigma)$. So we have increasing number of pairwise disjoint balls centered in $T_D(\Sigma) \subset M_i$ and thus$T_D(\Sigma) \subset M_i$ have no subsequences converging in the Gromov-Hausdorff sense [14].

□

6. Conjectures and Open Problems

We now consider the general case of complete asymptotically flat manifolds with nonnegative scalar curvature. We will restrict to dimension three, because we have the most tools available in dimension three. We consider whether the Positive Mass Theorem is stable in this setting:

Definition 6.1. *Let $\mathcal{M}$ be a subclass of asymptotically flat three dimensional Riemannian manifolds with nonnegative scalar curvature and no interior closed minimal surfaces and either no boundary or the boundary is an outermost minimizing surface.*

Conjecture 6.2. *Given any $\epsilon > 0$, $D > 0$, $\alpha_0 > 0$, there exists a $\delta = \delta(\epsilon, D, \alpha_0) > 0$ such that if $M^3 \in \mathcal{M}$ has ADM mass $\mathrm{m}_{\mathrm{ADM}}(M) < \delta$ and $\mathbb{E}^3$ is Euclidean space. Then*

$$d_{\mathcal{F}}(\, T_D(\Sigma_{\alpha_0}) \subset M^3\, ,\, T_D(\Sigma_{\alpha_0}) \subset \mathbb{E}^3\,) \,<\, \epsilon. \tag{244}$$

where Σ_{α_0} is a special surface *of area* $\mathrm{Vol}_2(\Sigma_{\alpha_0}) = \alpha_0$*, and $T_D(\Sigma_{\alpha_0})$ is the tubular neighborhood of radius D around Σ_{α_0}.*

We are deliberately vague as to the strength of our condition of *asymptotical flatness* in the definition of $\mathcal{M}$. The conjectures may require strong conditions at infinity. We have also been vague as to what the *special surface*, Σ, should be. We know the special surface must somehow avoid wells but also be uniquely defined in Euclidean space up to isometry. We provide possible choices for the strength of the asymptotic flatness and special surface in the following remarks.

Remark 6.3. Another possible choice of special surface, Σ, is a Constant Mean Curvature surface. One could say Σ achieves an isoperimetric condition: the surface enclosing the maximal volume for its given area α_0. Note in Bray's thesis it is proven that such a Σ exists if it is connected [4]. One could for example assume that the manifold has a smooth CMC foliation down to Σ with area α_0. Or one could just assume a smooth CMC foliation exists on $T_D(\Sigma)$ where Σ is a leaf in the foliation with no such strong assumption at infinity. There has been significant work on the existence of CMC foliations and their properties beginning with Huisken-Yau [16].

Remark 6.4. A stronger condition on Σ which might be viewed as a test case for the prior remark would be to require positive Gauss curvature or possibly even lying in a foliation of such surfaces. Nirenberg proved that such Σ isometrically embed into Euclidean space [24] which we have shown provides a metric isometric embedding in Theorem 3.2. Such surfaces have a well defined quasi-local mass defined by Liu-Yau [22] based on work of Shi-Tam [28] which would be controlled by the ADM mass at infinity.

Remark 6.5. A possible choice of special surface, Σ, is that it be a level set of Inverse Mean Curvature Flow from a point or from the boundary of M. One might assume the manifold has a smooth IMCF in the conjecture or one might assume only that the IMCF is smooth on a neighborhood containing $T_D(\Sigma)$. Geroch proved that smooth IMCF has a monotone Hawking mass [10], so it should be possible to control the metric in a way somewhat similar to the way in which we proved that monotonicty of the Hawking mass to provided Lipschitz controls on our rotationally symmetric metrics.

Remark 6.6. Huisken-Ilmanen extended the IMCF using Geometric Measure Theory to prove the Penrose Conjecture (and reprove the Positive Mass Theorem) [15]. Their proof uses a weak Inverse Mean Curvature Flow with a monotone quasilocal mass. Conjecture 6.2 might hold on any manifold satisfying the conditions of their theorem where Σ is a level set of their flow. Many difficulties would arise when trying to prove this. Since weak IMCF jumps over regions likes wells, one would need to control the volumes of those regions separately.

Remark 6.7. *One might consider the case where M^3 is a Spin manifold and apply the work of Finster* [8]. *Finster bounds the areas of level sets of spinors and controls the L^2 norms of the curvature tensor. It is possible that level sets of spinors provide an appropriate choice for the special surface Σ_0 although we have not investigated this closely.*

Remark 6.8. One might consider the case where M^m is a graph in Euclidean space. Here one could examine the situation with many wells and explicitly cut them out. One could apply Theorem 3.6 directly to find a filling manifold. In the graph setting one might test out various conditions at infinity and choices of special surface Σ perhaps even using numerical methods to solve IMCF and find CMC surfaces. Lam has provided a new short proof of the Positive Mass Theorem in the graph setting which may prove useful to those attempting to prove the conjecture in this case [18].

One may also consider the stability of the Penrose Inequality. The authors have completed an investigation of this in [20]. In fact, the Penrose Inequality is not even stable in the rotationaly symmetric case. However sequences of manifolds approaching equality in the Penrose inequality do have subsequences which converge in the pointed intrinsic flat sense to manifolds which are Schwarscshild spaces outside their outermost minimal surface. In fact, far stronger convergence can be obtained as there are no thin central wells just deep horizon central horizons which the authors prove converge to cylinders of various lengths in the Lipschitz sense.

The authors also prove Lipschitz convergence outside of the central well in the Positive Mass setting in that paper. Without rotational symmetry, the authors provide an example with increasingly dense thin deep wells much like the example in this paper [20]. Thus one expects at best pointed intrinsic flat convergence without rotational symmetry for almost equality of the Penrose Equality.

We close this paper with a call for the investigation of a scalable version of the Intrinsic Flat Distance.

Remark 6.9. Recall that the Intrinsic Flat Distance is the sum of a volume and an area in (1). This is a consequence of the fact that the Intrinsic Flat Distance defined in [30] is based on the flat distance of Federer-Fleming [7] which is a norm:

$$d_F(T_1, T_2) = |T_1 - T_2|_\flat = \inf\{M_m(A) + M_{m+1}(B) : A + \partial B = T_1 - T_2\}. \tag{245}$$

One may immediately consider a related scalable intrinsic flat distance which abandons the norm properties in favor of scalability so that

$$d_{sF}(T_1, T_2) = \inf\left\{M_m(A)^{1/m} + M_{m+1}(B)^{1/(m+1)} : A + \partial B = T_1 - T_2\right\}. \tag{246}$$

This is still a distance since it is nonnegative, symmetric, satisfies the triangle inequality and

$$d_{sF}(T_1, T_2) = 0 \iff d_F(T_1, T_2) = 0 \iff T_1 = T_2. \tag{247}$$

This can be seen by taking A_i, B_i with $A_i + \partial B_i = T_1 - T_2$ approaching the infimum and observing that $M(A_i), M(B_i) \to 0$ since masses of integral currents are nonnegative.

Thus one might consider defining an intrinsic scalable flat distance, $d_{s\mathcal{F}}$ between Riemannian manifolds such that

$$d_{s\mathcal{F}}(M_1^m, M_2^m) \le \mathrm{Vol}_{m+1}\left(B^{m+1}\right)^{1/(m+1)} + \mathrm{Vol}_m\left(A^m\right)^{1/m} \tag{248}$$

much as in [30] and investigating which theorems hold as they stand and which need adapting. This investigation would involve looking deeper than just this paper as the norm properties were applied on more than one occasion and in citations.

See Remark 4.6 for information about estimating this scalable flat distance in the rotationally symmetric case of the almost inequality in the Positive Mass Theorem.

While this final remark suggests a problem which would involve a strong understanding of geometric measure theory, we believe other problems suggested in this paper are really questions of geometric analysis.

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